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# DISPERSIVE LIMIT FROM THE KAWAHARA TO THE KDV EQUATION

LUC MOLINET AND YUZHAO WANG

ABSTRACT. We investigate the limit behavior of the solutions to the Kawahara equation

$$u_t + u_{3x} + \varepsilon u_{5x} + uu_x = 0, \quad \varepsilon > 0$$

as  $\varepsilon \rightarrow 0$ . In this equation, the terms  $u_{3x}$  and  $\varepsilon u_{5x}$  do compete together and do cancel each other at frequencies of order  $1/\sqrt{\varepsilon}$ . This prohibits the use of a standard dispersive approach for this problem. Nevertheless, by combining different dispersive approaches according to the range of spaces frequencies, we succeed in proving that the solutions to this equation converges in  $C([0, T]; H^1(\mathbb{R}))$  towards the solutions of the KdV equation for any fixed  $T > 0$ .

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Introduction.** In this paper we are interested in the limit behavior of the solutions to the Kawahara equation

$$(K_\varepsilon) \quad u_t + u_{3x} + \varepsilon u_{5x} + uu_x = 0, \quad (t, x) \in \mathbb{R}^2, \quad \varepsilon > 0,$$

as the positive coefficient  $\varepsilon \rightarrow 0$ .

Our goal is to prove that they converge in a strong sense towards the solutions of the KdV equation

$$(1.1) \quad u_t + u_{3x} + uu_x = 0, \quad (t, x) \in \mathbb{R}^2.$$

This study can be seen as a peculiar case of the following class of limit behavior problems :

$$(1.2) \quad \partial_t u + \partial_x (L_1 - \varepsilon L_2)u + N_1(u) + \varepsilon N_2(u) = 0,$$

where  $u : \mathbb{R} \rightarrow \mathbb{R}$ ,  $L_1$  and  $L_2$  are pseudo-differential operators with Fourier symbols  $|\xi|^{\alpha_1}$  and  $|\xi|^{\alpha_2}$  with  $0 < \alpha_1 < \alpha_2$  and  $N_1$  and  $N_2$  are polynomial functions that depends on  $u$ , its derivatives and possibly on the image of  $u$  by some pseudo-differential operator (as for instance the Hilbert transform) . Note that the dispersive limits from the Benjamin equation or some higher-order BO equations derived in [3] towards the Benjamin-Ono equation enter this class.

In this class of limit behavior problems, the main difficulty comes from the fact that the dispersive terms  $\partial_x L_1 u$  and  $\varepsilon \partial_x L_2 u$  do compete together. As one can easily check, the derivatives of the associated phase function  $\phi(\xi) = \xi|\xi|^{\alpha_1}(1 - \varepsilon|\xi|^{\alpha_2 - \alpha_1})$  does vanish at frequencies of order  $\varepsilon^{-\frac{1}{\alpha_2 - \alpha_1}}$ . This will make classical dispersive estimates as Strichartz estimates, global Kato smoothing effect or maximal in time

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estimate, not uniform in  $\varepsilon$ . Therefore it is not clear to get even boundedness uniformly in  $\varepsilon$  of the solutions to (1.2) by classical dispersive resolution methods.

On the other hand, by using only energy estimates that do not take into account the dispersive terms, we can see immediately that the solutions to  $(K_\varepsilon)$  will stay bounded in  $H^s(\mathbb{R})$ , uniformly in  $\varepsilon$ , providing we work in Sobolev spaces  $H^s(\mathbb{R})$  with index  $s > 3/2$ . Moreover, using for instance Bona-Smith argument, we could prove the convergence of the solution of  $(K_\varepsilon)$  to the ones of (1.1) in  $C([0, T]; H^s(\mathbb{R}))$  with  $T = T(\|u(0)\|_{H^s})$  and  $s > 3/2$ . However this approach is far to be satisfactory since it does not use at all the dispersive effects. Moreover, the KdV and Kawahara equations are known to be well-posed in low indices Sobolev spaces (see for instance [1], [8], [6]) and one can ask whether such convergence result does hold in those spaces. In this work we make a first step in this direction by proving that this convergence result holds in  $H^s(\mathbb{R})$  with  $s \geq 1$ . Note that  $H^1(\mathbb{R})$  is a natural space for this problem since it is the energy space for the KdV equation. Our main idea is to combine different dispersive method according to the area of frequencies we consider. More precisely, we will use a Bourgain's approach (cf. [1], [4]) outside the area  $D_\varepsilon$  where the first derivative of the phase function  $\phi'$  does vanish whereas we will use Koch-Tzvetkov approach (cf. [10]) in  $D_\varepsilon$ . Indeed, noticing that  $\phi''$  does not vanish in this area, the Strichartz estimate are valid uniformly in  $\varepsilon$  on  $D_\varepsilon$  so that we can apply Koch-Tzvetkov approach. On the other hand, outside  $D_\varepsilon$  one can easily see that one has a strong resonance relation at least for the worst interactions, namely the high-low interactions. Indeed, assuming that  $|\xi_1| \gg |\xi_2|$ , by the mean-value theorem, it holds

$$|\phi_\varepsilon(\xi_1 + \xi_2) - \phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_2)| \sim |\phi'_\varepsilon(\xi_1)\xi_2 - \phi_\varepsilon(\xi_2)| \sim |\phi'_\varepsilon(\xi_1)\xi_2| \sim |\xi^2(3 - 5\varepsilon\xi^2)\xi_2| \gtrsim \xi^2|\xi_2|,$$

where  $\xi = \xi_1 + \xi_2$  is the output frequency and  $\phi_\varepsilon(\xi) = \xi^3 - \varepsilon\xi^5$  is the phase function associated with the  $(K_\varepsilon)$ . It is worth noticing that this resonance relation is similar to the one of the KdV equation that reads  $(\xi_1 + \xi_2)^3 - (\xi_1)^3 - (\xi_2)^3 = 3\xi\xi_1\xi_2$ . To rely on this strong resonance relation even when one of the input frequency belongs to  $D_\varepsilon$  we will make use of the fact that any  $H^1$ -solution to  $(K_\varepsilon)$  must belong to some Bourgain's space with time regularity one.

## 1.2. Main results.

**Theorem 1.1.** *Let  $s \geq 1$ ,  $\varphi \in H^s(\mathbb{R})$ ,  $T > 0$  and  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  be a decreasing sequence of real numbers converging to 0. The sequence  $u_n \in C(\mathbb{R}; H^s(\mathbb{R}))$  of solutions to  $(K_\varepsilon)$  emanating from  $\varphi$  satisfies*

$$(1.3) \quad u_n \rightarrow u \text{ in } C([0, T]; H^s(\mathbb{R}))$$

where  $u \in C(\mathbb{R}; H^s(\mathbb{R}))$  is the unique solution to the KdV equation (1.1) emanating from  $\varphi$ .

Theorem 1 is actually a direct consequence of the fact that the Cauchy problem associated with  $(K_\varepsilon)$  is well-posed in  $H^s(\mathbb{R})$ ,  $s \geq 1$ , uniformly in  $\varepsilon \in ]0, 1[$  in the following sense

**Theorem 1.2.** *Let  $s \geq 1$  and  $\varphi \in H^s(\mathbb{R})$ . There exists  $T = T(\|\varphi\|_{H^1}) \in ]0, 1[$  and  $C > 0$  such that for any  $\varepsilon \in ]0, 1[$  the solution  $u_\varepsilon \in C(\mathbb{R}; H^1(\mathbb{R}))$  to  $(K_\varepsilon)$  satisfies*

$$(1.4) \quad \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s} \leq C\|\varphi\|_{H^s}$$

Moreover, for any  $R > 0$ , the family of solution-maps  $S_{K_\varepsilon} : \varphi \mapsto u_\varepsilon$ ,  $\varepsilon \in ]0, 1[$ , from  $B(0, R)_{H^s}$  into  $C([0, T(R)]; H^s(\mathbb{R}))$  is equi-continuous, i.e. for any sequence  $\{\varphi_n\} \subset B(0, R)_{H^s}$  converging to  $\varphi$  in  $H^s(\mathbb{R})$  it holds

$$(1.5) \quad \lim_{n \rightarrow 0} \sup_{\varepsilon \in ]0, 1[} \|S_{K_\varepsilon} \varphi - S_{K_\varepsilon} \varphi_n\|_{L^\infty(0, T(R); H^s(\mathbb{R}))} = 0.$$

**1.3. Notation.** For any positive numbers  $a$  and  $b$ , the notation  $a \lesssim b$  means that there exists a positive constant  $c$  such that  $a \leq cb$ . We also denote  $a \sim b$  when  $a \lesssim b$  and  $b \lesssim a$ . Moreover, if  $\alpha \in \mathbb{R}$ ,  $\alpha_+$ , respectively  $\alpha_-$ , will denote a number slightly greater, respectively lesser, than  $\alpha$ .

For  $u = u(x, t) \in \mathcal{S}(\mathbb{R}^2)$ ,  $\mathcal{F}u = \hat{u}$  will denote its space-time Fourier transform, whereas  $\mathcal{F}_x u = (u)^\wedge_x$ , respectively  $\mathcal{F}_t u = (u)^\wedge_t$ , will denote its Fourier transform in space, respectively in time. For  $s \in \mathbb{R}$ , we define the Bessel and Riesz potentials  $J_x^s$  and  $D_x^s$ , by

$$J_x^s u = \mathcal{F}_x^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}_x u) \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}(|\xi|^s \mathcal{F}_x u).$$

We will need a Littlewood-Paley analysis. Let  $\psi \in C_0^\infty(\mathbb{R})$  be an even function such that  $\psi \geq 0$ ,  $\text{supp } \psi \subset [-3/2, 3/2]$ ,  $\psi \equiv 1$  on  $[-5/4, 5/4]$ . We set  $\eta_0 := \psi$  and for all  $k \in \mathbb{N}^*$ ,  $\eta_{2^k}(\xi) := \psi(2^{-k}\xi) - \psi(2^{-k+1}\xi)$ ,  $\eta_{\leq 2^k} := \psi(2^{-k}\cdot) = \sum_{j=0}^k \eta_{2^j}$  and  $\eta_{\geq 2^k} := 1 - \psi(2^{k-1}\cdot) = 1 - \eta_{\leq 2^{k-1}}$ . The Fourier multiplier operators by  $\eta_{2^j}$ ,  $\eta_{\leq 2^j}$  and  $\eta_{\geq 2^j}$  will be denoted respectively by  $P_{2^j}$ ,  $P_{\leq 2^j}$  and  $P_{\geq 2^j}$ , i.e. for any  $u \in L^2(\mathbb{R})$

$$\widehat{P_{2^j} u} := \eta_{2^j} \hat{u}, \quad \widehat{P_{\leq 2^j} u} := \eta_{\leq 2^j} \hat{u} \quad \text{and} \quad \widehat{P_{\geq 2^j} u} := \eta_{\geq 2^j} \hat{u}.$$

Note that, to simplify the notations, any summations over capitalized variables such as  $N$  are presumed to be dyadic with  $N \geq 1$ , i.e., these variables range over numbers of the form  $2^k$ ,  $k \in \mathbb{Z}_+$ .  $P_+$  and  $P_-$  will denote the projection on respectively the positive and the negative Fourier frequencies.

Finally, we denote by  $U_\varepsilon(t) := e^{-t(\partial_x^3 + \varepsilon \partial_x^5)}$  the free evolution associated with the linear part of  $(K_\varepsilon)$ .

**1.4. Function spaces.** For  $1 \leq p \leq \infty$ ,  $L^p(\mathbb{R})$  is the usual Lebesgue space with the norm  $\|\cdot\|_{L^p}$ , and for  $s \in \mathbb{R}$ , the real-valued Sobolev spaces  $H^s(\mathbb{R})$  denote the spaces of all real-valued functions with the usual norms

$$\|\varphi\|_{H^s} = \|J_x^s \varphi\|_{L^2}.$$

If  $f = f(x, t)$  is a function defined for  $x \in \mathbb{R}$  and  $t$  in the time interval  $[0, T]$ , with  $T > 0$ , if  $B$  is one of the spaces defined above,  $1 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ , we will define the mixed space-time spaces  $L_T^p B_x$ ,  $L_t^p B_x$ ,  $L_x^q L_T^p$  by the norms

$$\|f\|_{L_T^p B_x} = \left( \int_0^T \|f(\cdot, t)\|_B^p dt \right)^{\frac{1}{p}}, \quad \|f\|_{L_t^p B_x} = \left( \int_{\mathbb{R}} \|f(\cdot, t)\|_B^p dt \right)^{\frac{1}{p}},$$

and

$$\|f\|_{L_x^q L_T^p} = \left( \int_{\mathbb{R}} \left( \int_0^T |f(x, t)|^p dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}.$$

For  $s, b \in \mathbb{R}$ , we introduce the Bourgain spaces  $X_\varepsilon^{s, b}$  related to the linear part of  $(K_\varepsilon)$  as the completion of the Schwartz space  $\mathcal{S}(\mathbb{R}^2)$  under the norm

$$(1.6) \quad \|v\|_{X_\varepsilon^{s, b}} := \left( \int_{\mathbb{R}^2} \langle \tau - \phi_\varepsilon(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\hat{v}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}},$$

where  $\langle x \rangle := 1 + |x|$ . We will also use a dyadic version of those spaces introduced in [11] in the context of wave maps. For  $s, b \in \mathbb{R}$ ,  $1 \leq q \leq \infty$ ,  $X_\epsilon^{s,b,q}$  will denote the completion of the Schwartz space  $\mathcal{S}(\mathbb{R}^2)$  under the norm

$$(1.7) \quad \|v\|_{X_\epsilon^{s,b,q}} := \left( \sum_{k \geq 0} \left( \sum_{j \geq 0} \langle 2^k \rangle^{sq} \langle 2^j \rangle^{bq} \|P_{2^k}(\xi) P_{2^j}(\tau - \phi_\epsilon(\xi)) \widehat{v}(\xi, \tau)\|_{L_{\tau, \xi}^2}^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}.$$

Moreover, we define a localized (in time) version of these spaces. Let  $T > 0$  be a positive time and  $Y = X_\epsilon^{s,b}$  or  $Y = X_\epsilon^{s,b,q}$ . Then, if  $v : \mathbb{R} \times ]0, T[ \rightarrow \mathbb{C}$ , we have that

$$\|v\|_{Y_T} := \inf \{ \|\tilde{v}\|_Y \mid \tilde{v} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}, \tilde{v}|_{\mathbb{R} \times ]0, T[} = v \}.$$

## 2. UNIFORM ESTIMATES FAR FROM THE STATIONARY POINT OF THE PHASE FUNCTION

As we explained in the introduction, it is crucial that the first and the second derivatives of the phase function  $\phi_\epsilon(\xi) = \xi^3 - \epsilon \xi^5$  do not cancel exactly at the same point. Indeed,  $\phi'_\epsilon(\xi) = 0 \Leftrightarrow |\xi| = \sqrt{\frac{3}{5\epsilon}}$  while  $\phi''_\epsilon(\xi) = 0 \Leftrightarrow |\xi| = \sqrt{\frac{3}{10\epsilon}}$ . Consequently, we introduce the following smooth Fourier projectors

$$\widehat{P_{A_\epsilon} f} = \left[ 1 - \eta_0 \left[ 20\sqrt{\epsilon} \left( |\xi| - \sqrt{\frac{3}{5\epsilon}} \right) \right] \right] \widehat{f}$$

and

$$\widehat{P_{B_\epsilon} f} = \left[ 1 - \eta_0 \left[ 20\sqrt{\epsilon} \left( |\xi| - \sqrt{\frac{3}{10\epsilon}} \right) \right] \right] \widehat{f}$$

Clearly,  $\widehat{P_{A_\epsilon} f}$  cancels in a region of order  $\epsilon^{-1/2}$  around  $\sqrt{\frac{3}{5\epsilon}}$  whereas  $\widehat{P_{B_\epsilon} f}$  cancels in a region of order  $\epsilon^{-1/2}$  around  $\sqrt{\frac{3}{10\epsilon}}$ . We are now in position to state the main proposition of this section :

**Proposition 2.1.** *Let  $s \geq 1$ ,  $0 < T < 1$  and  $u_{i,\epsilon} \in C([0, T]; H^s(\mathbb{R}))$ ,  $i = 1, 2$ , be two solutions to  $(K_\epsilon)$  with  $0 < \epsilon \ll 1$  and initial data  $\varphi_i$ . Then it holds*

$$(2.1) \quad \|P_{A_\epsilon} u_{i,\epsilon}\|_{X_{\epsilon,T}^{s,1/2,1}} \lesssim \|\varphi_i\|_{H^s} + T^{1/4} \|u_{i,\epsilon}\|_{Y_{\epsilon,T}^s} \|u_{i,\epsilon}\|_{Y_{\epsilon,T}^1} (1 + \|u_{i,\epsilon}\|_{Y_{\epsilon,T}^1})$$

and, setting  $w = u_{1,\epsilon} - u_{2,\epsilon}$ ,

$$(2.2) \quad \|P_{A_\epsilon} w\|_{X_{\epsilon,T}^{s,1/2,1}} \lesssim \|\varphi_1 - \varphi_2\|_{H^s} + T^{1/4} \|w\|_{Y_{\epsilon,T}^s} \sum_{i=1}^2 \|u_{i,\epsilon}\|_{Y_{\epsilon,T}^s} (1 + \|u_{i,\epsilon}\|_{Y_{\epsilon,T}^s})$$

where

$$(2.3) \quad \|u\|_{Y_{\epsilon,T}^s} := \|P_{A_\epsilon} u\|_{X_{\epsilon,T}^{s,1/2,1}} + \|u\|_{L_T^\infty H^s}$$

We will make a frequent use of the following linear estimates

**Lemma 2.1.** *Let  $\varphi \in \mathcal{S}(\mathbb{R})$  and  $T \in ]0, 1]$  then  $\forall 0 < \epsilon \ll 1$ ,*

$$(2.4) \quad \|P_{A_\epsilon} \partial_x U_\epsilon(t) \varphi\|_{L_x^\infty L_t^2} \lesssim \|\varphi\|_{L^2}$$

$$(2.5) \quad \|D_x^{1/4} P_{B_\epsilon} U_\epsilon(t) \varphi\|_{L_t^4 L_x^\infty} \lesssim \|\varphi\|_{L^2}$$

$$(2.6) \quad \|P_{\leq 2} U_\epsilon(t) \varphi\|_{L_x^2 L_T^\infty} \lesssim \|\varphi\|_{L^2},$$

where  $\mathcal{F}_x(P_{A_\epsilon} \varphi) = (1 - \eta_{A_\epsilon}) \mathcal{F}_x \varphi$  and the implicit constants are independent of  $\epsilon > 0$ .

*Proof.* First, (2.4) follows from the classical proof of the local Kato smoothing effect, by using that  $|\phi'_\varepsilon(\xi)| \gtrsim |\xi|^2$  on the Fourier support of  $P_{A_\varepsilon}$ .

To prove (2.5), we first notice that the Fourier support of  $P_{B_\varepsilon}$  does not intersect the region  $\{\xi \in \mathbb{R}, |\xi| \in [\sqrt{\frac{1}{4\varepsilon}}, \sqrt{\frac{7}{20\varepsilon}}]\}$ . By the  $TT^*$  argument it suffices to prove that

$$(2.7) \quad \|U_\varepsilon(t)D_x^{1/2}P_{B_\varepsilon}\varphi\|_{L_x^\infty} + \|U_\varepsilon(t)D_x^{1/2}P_{A_\varepsilon}\varphi\|_{L_x^\infty} \lesssim t^{-1/2}\|\varphi\|_{L^1}$$

By classical arguments, (1.3) will be proven if we show

$$\left\| \int_{\mathbb{R}} \chi_{\{|\xi| \notin [\sqrt{\frac{1}{4\varepsilon}}, \sqrt{\frac{7}{20\varepsilon}}]\}} |\xi|^{1/2} e^{i[x\xi + (\xi^3 - \varepsilon\xi^5)t]} d\xi \right\|_{L_x^\infty} \lesssim t^{-1/2}.$$

Setting  $\theta := \xi|t|^{1/3}$  this is equivalent to prove

$$(2.8) \quad I_\varepsilon := \sup_{t \in \mathbb{R}, X \in \mathbb{R}} \left| \int_{\mathbb{R}} \chi_{\{|\theta| \notin [\sqrt{\frac{|t|^{2/3}}{4\varepsilon}}, \sqrt{\frac{7|t|^{2/3}}{20\varepsilon}}]\}} |\theta|^{1/2} e^{i[X\theta + \theta^3 - \frac{\varepsilon}{|t|^{2/3}}\theta^5]} d\theta \right| \lesssim 1$$

We set  $\Phi(\theta) = \Phi_{t,\varepsilon}(\theta) := \theta^3 - \frac{\varepsilon}{|t|^{2/3}}\theta^5$  and notice that

$$\Phi'(\theta) := 3\theta^2 - \frac{5\varepsilon}{|t|^{2/3}}\theta^4 \quad \text{and} \quad \Phi''(\theta) = 2\theta\left(3 - \frac{10\varepsilon}{|t|^{2/3}}\theta^2\right).$$

(2.8) is obvious when restricted on  $|\theta| \leq 100$ . Now, it is worth noticing that

$$|\Phi''(\theta)| \gtrsim 1 + \max\left(|\theta|, \frac{\varepsilon}{|t|^{2/3}}\theta^3\right)$$

whenever  $\theta \in \{|\theta| \geq 100 / |\theta| \notin [\sqrt{\frac{|t|^{2/3}}{4\varepsilon}}, \sqrt{\frac{7|t|^{2/3}}{20\varepsilon}}]\}$ . Therefore, in the region  $|\theta| \in [\sqrt{\frac{|t|^{2/3}}{10\varepsilon}}, \sqrt{\frac{2|t|^{2/3}}{\varepsilon}}]$ , (2.8) follows from Van der Corput lemma since  $|\Phi''(\theta)| \gtrsim 1 + \frac{|t|^{1/3}}{\sqrt{\varepsilon}}$  and  $|\theta|^{1/2} \sim \frac{|t|^{1/6}}{\varepsilon^{1/4}}$ . It thus remains to consider the region  $|\theta| \notin [\sqrt{\frac{|t|^{2/3}}{10\varepsilon}}, \sqrt{\frac{2|t|^{2/3}}{\varepsilon}}]$ . We notice that, in this region, it holds

$$(2.9) \quad |\Phi'(\theta)| \sim |\theta|^2 \text{ for } |\theta| \leq \sqrt{\frac{|t|^{2/3}}{10\varepsilon}} \quad \text{and} \quad |\Phi'(\theta)| \sim \frac{\varepsilon|\theta|^4}{|t|^{2/3}} \text{ for } |\theta| \geq \sqrt{\frac{2|t|^{2/3}}{\varepsilon}}$$

and divide this region into two subregions.

- The subregion  $|\Phi'(\theta) - X| \leq |X|/2$ . Then  $|\Phi'(\theta)| \sim |X|$ . Assuming we are in the region  $100 < |\theta| \leq \sqrt{\frac{|t|^{2/3}}{10\varepsilon}}$ , we have  $|\Phi'(\theta)| \sim |\theta|^2$  and thus  $|\theta| \sim \sqrt{|X|}$ . Then (2.8) follows from Van der Corput lemma since  $|\Phi''(\theta)| \gtrsim |\theta| \sim \sqrt{|X|}$ . On the other hand, assuming that  $|\theta| \geq \sqrt{\frac{2|t|^{2/3}}{\varepsilon}} \geq 100$  then  $|\Phi'(\theta)| \sim \varepsilon|\theta|^4|t|^{-2/3}$  and thus  $|\theta| \sim \varepsilon^{-1/4}|X|^{1/4}|t|^{1/6}$ . (2.8) follows again from Van der Corput lemma since  $|\Phi''(\theta)| \gtrsim |\theta| \sim \varepsilon^{-1/4}|X|^{1/4}|t|^{1/6}$ .

- The subregion  $|\Phi'(\theta) - X| > |X|/2$ . Then  $|\Phi'(\theta) - X| \sim |\Phi'(\theta)|$  and (2.8) is obtained by integrating by parts and using (2.9). This completes the proof of (2.5).

Finally, to show (2.6) we notice that it suffices to prove that for  $|x| \geq 10^4$ ,

$$\sup_{t \in [0,1]} \left| \int_{\mathbb{R}} \eta_{\leq 2}(\xi) e^{i[x\xi + \phi_\varepsilon(\xi)t]} d\xi \right| \lesssim |x|^{-2},$$

where  $\phi_\varepsilon(\xi) = \xi^3 - \varepsilon\xi^5$ . But this follows directly by integrating by parts twice since  $|x - \phi'_\varepsilon(\xi)t| \gtrsim |x|$  for any  $|t| \leq 1$  and  $|\xi| \leq 4$ .  $\square$

To prove Proposition 2.1 we will have to put the whole solution  $u_\varepsilon$  of  $(K_\varepsilon)$  and not only  $P_{A_\varepsilon}u_\varepsilon$  in some Bourgain's space with regularity 1 in time. This will be done in the next lemma by noticing that any solution to  $(K_\varepsilon)$  that belongs to  $C([0, T]; H^1(\mathbb{R}))$  automatically belongs to  $X_{\varepsilon, T}^{0,1}$ .

**Lemma 2.2.** *Let  $s \geq 1$ ,  $T \in ]0, 1[$  and  $u \in C([0, T]; H^s(\mathbb{R}))$  be a solution to  $(K_\varepsilon)$ . Then,*

$$(2.10) \quad \|u\|_{X_{\varepsilon, T}^{s-1,1}} \lesssim \|u\|_{L_T^\infty H_x^{s-1}} + \|u\|_{L_T^\infty H_x^1} \|u\|_{L_T^\infty H_x^s},$$

where the implicit constant is independent of  $\varepsilon$ .

*Proof.* First, we consider  $v(t) = U_\varepsilon(-t)u(t)$  on the time interval  $]0, T]$  and extend  $v$  on  $] -2, 2[$  by setting  $\partial_t v = 0$  on  $[-2, 2] \setminus [0, T]$ . Then, it is pretty clear that

$$\|\partial_t v\|_{L^2([-2, 2]; H_x^{s-1})} = \|\partial_t v\|_{L_T^2 H_x^{s-1}}, \quad \text{and} \quad \|v\|_{L^2([-2, 2]; H_x^{s-1})} \lesssim \|v\|_{L_T^\infty H_x^{s-1}}.$$

Now, we define  $\tilde{u}(x, t) = \eta(t)U(t)v(t)$ . Obviously,  $\tilde{u}$  is an extension of  $u$  outside  $] -T, T[$  and it holds

$$(2.11) \quad \|\tilde{u}\|_{X_{\varepsilon}^{s-1,1}} \lesssim \|\partial_t v\|_{L^2([-2, 2]; H_x^{s-1})} + \|v\|_{L^2([-2, 2]; H_x^{s-1})} \lesssim \|\partial_t v\|_{L_T^2 H_x^{s-1}} + \|v\|_{L_T^\infty H_x^{s-1}}.$$

Therefore (2.10) follows from the identity

$$\partial_t v = U_\varepsilon(-t) \left[ u_t + u_{xxx} + \varepsilon u_{5x} \right]$$

together with the facts that  $u$  is a solution to  $(K_\varepsilon)$  and that

$$\|uu_x\|_{H_x^{s-1}} \leq \|u^2\|_{H_x^s} \lesssim \|u\|_{L_x^\infty} \|u\|_{H_x^s}$$

as soon as  $s \geq 1$ . □

Now, according to the Duhamel formula and to classical linear estimates in Bourgain's spaces (cf. [1], [4]), Proposition 2.1 is a direct consequence of the following bilinear estimate

$$(2.12) \quad \|P_{A_\varepsilon} \partial_x(u_1 u_2)\|_{X_{\varepsilon}^{s, -1/2, 1}} \lesssim T^{1/4} \left( \|u_1\|_{Y_\varepsilon^s} + \|u_1\|_{X_{\varepsilon}^{s-1, 1}} \right) \left( \|u_2\|_{Y_\varepsilon^1} + \|u_2\|_{X_{\varepsilon}^{0, 1}} \right) \\ + T^{1/4} \left( \|u_1\|_{Y_\varepsilon^1} + \|u_1\|_{X_{\varepsilon}^{0, 1}} \right) \left( \|u_2\|_{Y_\varepsilon^{s-1}} + \|u_2\|_{X_{\varepsilon}^{s-1, 1}} \right),$$

where the functions  $u_i$  are supported in time in  $] -T, T[$  with  $0 < T \leq 1$ . To prove this bilinear estimate we first note that by symmetry it suffices to consider  $\partial_x \Lambda(u, v)$  where  $\Lambda(\cdot, \cdot)$  is defined by

$$\mathcal{F}_x(\Lambda(u, v)) := \int_{\mathbb{R}} \chi_{|\xi_1| \leq |\xi - \xi_1|} (\mathcal{F}_x u)(\xi_1) (\mathcal{F}_x v)(\xi - \xi_1) d\xi_1.$$

Moreover, using that for any  $s \geq 1$ ,

$$\langle \xi_1 + \xi_2 \rangle^s \lesssim \langle \xi_1 + \xi_2 \rangle \left( \langle \xi_1 \rangle^{s-1} + \langle \xi_2 \rangle^{s-1} \right),$$

it is a classical fact that we can restrict ourself to prove (2.12) for  $s = 1$ .

As mentioned in the introduction, the following resonance relation is crucial for our analysis in this frequency area :

$$(2.13) \quad \Theta(\xi, \xi_1) := \sigma - \sigma_1 - \sigma_2 = \xi \xi_1 (\xi - \xi_1) \left[ 3 - 5\varepsilon \left( (\xi_1 + \xi_2)^2 - \xi_1 \xi_2 \right) \right]$$

where

$$\sigma := \sigma(\tau, \xi) := \tau - \xi^3 - \varepsilon \xi^5, \quad \sigma_1 := \sigma(\tau_1, \xi_1) \quad \text{and} \quad \sigma_2 := \sigma(\tau - \tau_1, \xi - \xi_1).$$

We start by noticing that the case of output frequencies of order less or equal to one is harmless. Indeed, it is easy to check that for any couple  $u_i$ ,  $i = 1, 2$ , of smooth functions supported in time in  $] - T, T[$  with  $0 < T \leq 1$  it holds

$$(2.14) \quad \|\partial_x P_{A_\varepsilon} P_{\leq 8} \Lambda(u_1, u_2)\|_{X_\varepsilon^{1, -1/2, 1}} \lesssim \|\Lambda(u_1, u_2)\|_{L^2} \lesssim \|u_1\|_{L_t^\infty H^1} \|u_2\|_{L_t^\infty H^1}.$$

Let us continue by deriving an estimate for the interactions of high frequencies with frequencies of order less or equal to 1.

**Lemma 2.3.** *Let  $u_i$ ,  $i = 1, 2$ , be two smooth functions supported in time in  $] - T, T[$  with  $0 < T \leq 1$ . Then it holds*

$$(2.15) \quad \|\partial_x P_{A_\varepsilon} \Lambda(P_{\leq 8} u_1, u_2)\|_{X_\varepsilon^{1, -1/2, 1}} \lesssim \|u_1\|_{X_\varepsilon^{0, 1}} \left( T^{1/4} (\|P_{A_\varepsilon} u_2\|_{X_\varepsilon^{1, 1/2, 1}} + \|u_2\|_{X_\varepsilon^{0, 1}}) + \|\partial_x u_2\|_{L_{tx}^2} \right).$$

*Proof.* Since the norms in the right-hand side of (2.15) only see the size of the modulus of the Fourier transform, we can assume that all our functions have non negative Fourier transform. We set  $\eta_{A_\varepsilon} = 1 - \eta_0 \left[ 20\sqrt{\varepsilon} \left( |\xi| - \sqrt{\frac{3}{5\varepsilon}} \right) \right]$  so that  $\widehat{P_{A_\varepsilon} f} = \eta_{A_\varepsilon} \widehat{f}$ . Rewriting  $\eta_{A_\varepsilon}(\xi)$  as  $\eta_{A_\varepsilon}(\xi - \xi_1) + (\eta_{A_\varepsilon}(\xi) - \eta_{A_\varepsilon}(\xi - \xi_1))$ , it suffices to estimate the two following terms

$$I_1 := \left\| \mathcal{F}_x^{-1} \left( \partial_x \Lambda(\eta_{\leq 8} \mathcal{F}_x(u_1), \eta_{A_\varepsilon} \mathcal{F}_x(u_2)) \right) \right\|_{X^{1, -1/2, 1}}$$

and

$$I_2 := \left\| \mathcal{F}_x^{-1} \left( \xi \int_{\mathbb{R}} \eta_{\leq 8}(\xi_1) \mathcal{F}_x(u_1)(\xi_1) (\eta_{A_\varepsilon}(\xi) - \eta_{A_\varepsilon}(\xi - \xi_1)) \mathcal{F}_x(u_2)(\xi - \xi_1) d\xi_1 \right) \right\|_{X^{1, -1/2, 1}}$$

$I_1$  is easily estimate thanks to (2.6) by

$$\begin{aligned} I_1^2 &\lesssim \sum_{N \geq 1} T^{\frac{1}{2}-} \left\| (\eta_{\leq 8} \widehat{u_1}) * (\eta_N \eta_{A_\varepsilon} \widehat{\partial_x^2 u_2}) \right\|_{L^2}^2 \\ &\lesssim T^{\frac{1}{2}-} \sum_{N \geq 1} \|P_{\leq 8} u_1\|_{L_x^2 L_t^\infty}^2 \|\partial_x^2 P_N P_{A_\varepsilon} u_2\|_{L_x^\infty L_t^2}^2 \\ &\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}}^2 \|P_{A_\varepsilon} u_2\|_{X^{1, 1/2, 1}}^2. \end{aligned}$$

To estimate  $I_2$  we first notice that for  $|\xi_1| \leq 4$  and  $0 < \varepsilon < 10^{-8}$ ,

$$(2.16) \quad \eta_{A_\varepsilon}(\xi) - \eta_{A_\varepsilon}(\xi - \xi_1) = 0 \text{ whenever } |\xi| \in \left[ \frac{15}{16} \sqrt{\frac{3}{5\varepsilon}}, \frac{17}{16} \sqrt{\frac{3}{5\varepsilon}} \right] \cup \mathbb{C} \left[ \frac{2^{-3}}{\sqrt{\varepsilon}}, \frac{2^3}{\sqrt{\varepsilon}} \right].$$

and for any  $(\xi, \xi_1) \in \mathbb{R}^2$ ,

$$(2.17) \quad |\eta_{A_\varepsilon}(\xi) - \eta_{A_\varepsilon}(\xi - \xi_1)| \lesssim \min(1, \sqrt{\varepsilon} |\xi_1|).$$

Moreover, in the region  $|\xi_1| \leq 4$  and  $|\xi| \notin [\frac{15}{16} \sqrt{\frac{3}{5\varepsilon}}, \frac{17}{16} \sqrt{\frac{3}{5\varepsilon}}]$  the resonance relation (2.13) ensures that

$$(2.18) \quad |\sigma_{max}| := \max(|\sigma|, |\sigma_1|, |\sigma_2|) \gtrsim |\xi \xi_1 (\xi - \xi_1)|$$

where  $\sigma(\tau, \xi) := \tau - \phi_\varepsilon(\xi)$ ,  $\sigma_1 = \sigma(\tau_1, \xi_1)$  and  $\sigma_2 = \sigma(\tau - \tau_1, \xi - \xi_1)$ . We separate three regions



- $\sigma_{max} = \sigma_2$ . Then according to (2.16)-(2.18),

$$\begin{aligned}
I_2 &\lesssim T^{\frac{1}{2}-} \left\| \int_{\mathbb{R}^2} (\eta_{\leq 8} \widehat{u_1})(\xi_1, \tau_1) \sqrt{\varepsilon} \frac{|\xi_1| \langle \xi \rangle^2}{|\xi_1| |\xi - \xi_1|^2} \langle \sigma_2 \rangle \chi_{\{|\xi - \xi_1| \sim \frac{1}{\sqrt{\varepsilon}}\}} \widehat{u_2}(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1 \right\|_{L_{\xi, \tau}^2(|\xi| \sim \frac{1}{\sqrt{\varepsilon}})} \\
&\lesssim T^{\frac{1}{2}-} \|P_{\leq 8} u_1\|_{L_{tx}^\infty} \|u_2\|_{X^{-1/2, 1}} \\
&\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}} \|u_2\|_{X^{0, 1}}
\end{aligned}$$

- $\sigma_{max} = \sigma_1$ . Then according to (2.16)-(2.18),

$$\begin{aligned}
I_2 &\lesssim T^{\frac{1}{2}-} \left\| \langle \xi \rangle^2 \int_{\mathbb{R}^2} \frac{\langle \sigma_1 \rangle}{|\xi_1| |\xi - \xi_1|^2} (\eta_{\leq 8} \widehat{u_1})(\xi_1, \tau_1) \sqrt{\varepsilon} |\xi_1| \chi_{\{|\xi - \xi_1| \sim \frac{1}{\sqrt{\varepsilon}}\}} \widehat{u_2}(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1 \right\|_{L_{\xi, \tau}^2(|\xi| \sim \frac{1}{\sqrt{\varepsilon}})} \\
&\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}} \|D_x^{-1/2} \mathcal{F}^{-1}(\chi_{\{|\xi| \sim \frac{1}{\sqrt{\varepsilon}}\}} \widehat{u_2})\|_{L_{tx}^\infty} \\
&\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}} \|\mathcal{F}^{-1}(\chi_{\{|\xi| \sim \frac{1}{\sqrt{\varepsilon}}\}} \widehat{u_2})\|_{L_t^\infty L_x^2} \\
&\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}} \|u_2\|_{X^{0, 1}}
\end{aligned}$$

- $\sigma_{max} = \sigma$ . Then according to (2.16)-(2.18),

$$\begin{aligned}
I_2 &\lesssim \left\| \langle \xi \rangle^2 \int_{\mathbb{R}} \frac{\sqrt{\varepsilon} |\xi_1|}{|\xi_1|^{3/8} |\xi - \xi_1|^{3/4}} (\eta_{\leq 8} \widehat{u_1})(\xi_1) \chi_{\{|\xi - \xi_1| \sim \frac{1}{\sqrt{\varepsilon}}\}} \widehat{u_2}(\xi - \xi_1) d\xi_1 \right\|_{L^2(|\xi| \sim \frac{1}{\sqrt{\varepsilon}})} \\
&\lesssim \sqrt{\varepsilon} \|P_{\leq 8} u_1\|_{L_{tx}^\infty} \|D_x^{5/4} \mathcal{F}^{-1}(\chi_{\{|\xi| \sim \frac{1}{\sqrt{\varepsilon}}\}} \widehat{u_2})\|_{L_{tx}^2} \\
&\lesssim \|u_1\|_{X^{0, 1}} \|\partial_x u_2\|_{L_{tx}^2}
\end{aligned}$$

This completes the proof of the lemma.  $\square$

The next lemma ensures that the restriction of the left-side member of (2.12) on the region  $|\xi| \gtrsim 1$ ,  $|\xi_1| \gtrsim 1$  and  $|\sigma_{max}| \geq 2^{-5} |\xi \xi_1 (\xi - \xi_1)|$  can be easily controlled.

**Lemma 2.4.** *Under the same hypotheses as in Lemma 2.3, in the region where the following strong resonance relation holds*

$$(2.19) \quad |\sigma_{max}| \geq 2^{-5} |\xi \xi_1 (\xi - \xi_1)| ,$$

we have

$$(2.20) \quad \|\partial_x P_{A_\varepsilon} P_{\geq 8} \Lambda(P_{\geq 8} u_1, u_2)\|_{X_\varepsilon^{1, -1/2, 1}} \lesssim T^{1/4} \|u_1\|_{X^{0, 1}} \|u_2\|_{X^{0, 1}} + \left( \|u_1\|_{X^{0, 1}} + \|\partial_x u_1\|_{L_{tx}^2} \right) \|\partial_x u_2\|_{L_{tx}^2} .$$

*Proof.* Again we notice that the norms in the right-hand side of (2.4) only see the size of the modulus of the Fourier transforms. We can thus assume that all our functions have non-negative Fourier transforms. We set  $I := \|\partial_x P_{A_\varepsilon} P_{\geq 8} \Lambda(P_{\geq 8} u_1, u_2)\|_{X_\varepsilon^{1, -1/2, 1}}$  and separate different subregions .

- $|\sigma_1| \geq 2^{-5} |\xi \xi_1 (\xi - \xi_1)|$ . Then direct calculations give

$$\begin{aligned}
I &\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}} \|D_x^{-1} P_{\geq 2} u_2\|_{L_{tx}^\infty} \\
&\lesssim T^{\frac{1}{2}-} \|u_1\|_{X^{0, 1}} \|u_2\|_{X^{0, 1}} .
\end{aligned}$$

- $|\sigma_2| \geq 2^{-5} |\xi \xi_1 (\xi - \xi_1)|$ . This case can be treated exactly in the same way by exchanging the role of  $u_1$  and  $u_2$ .
- $|\sigma| \geq 2^{-5} |\xi \xi_1 (\xi - \xi_1)|$  and  $\max(|\sigma_1|, |\sigma_2|) \leq 2^{-5} |\xi \xi_1 (\xi - \xi_1)|$ .

Then we separate two subregions.

1.  $|\xi_1| \geq 2^{-7}|\xi|$ . Then  $|\xi_1| \gtrsim |\xi_{max}|$  and taking  $\delta > 0$  close enough to 0 we get

$$\begin{aligned} I &\lesssim \|\partial_x P_{A_\varepsilon} P_{\geq 8} \Lambda(P_{\geq 8} u_1, u_2)\|_{X_\varepsilon^{1, -1/2+\delta}} \\ &\lesssim \left\| \partial_x u_2 D_x^{-1/2+3\delta} P_{\geq 8} u_1 \right\|_{L^2} \\ &\lesssim \|D_x^{-1/2+3\delta} P_{\geq 8} u_1\|_{L_{tx}^\infty} \|\partial_x u_2\|_{L_{tx}^2} \\ &\lesssim \|u_1\|_{X^{1/4, 3/4}} \|\partial_x u_2\|_{L_{tx}^2} \\ &\lesssim (\|u_1\|_{X^{0,1}} + \|\partial_x u_1\|_{L_{tx}^2}) \|\partial_x u_2\|_{L_{tx}^2}. \end{aligned}$$

2.  $|\xi_1| \leq 2^{-7}|\xi|$ . Then, we notice that in this region  $\frac{1}{2}|\xi| \leq |\xi - \xi_1| \leq 2|\xi|$  and thus

$$(1 - 2^{-6})\xi^2 \leq \xi^2 - \xi_1(\xi - \xi_1) \leq (1 + 2^{-6})\xi^2.$$

Since  $\eta_{A_\varepsilon}$  does vanish on  $\left\{|\xi| \in \left[\frac{15}{16}\sqrt{\frac{3}{5\varepsilon}}, \frac{17}{16}\sqrt{\frac{3}{5\varepsilon}}\right]\right\}$ , we deduce from (2.13) that

$$(2.21) \quad |\sigma| \sim \max\left(|\xi\xi_1(\xi - \xi_1)|, \varepsilon|\xi^3\xi_1(\xi - \xi_1)|\right)$$

on the support of  $\eta_{A_\varepsilon}$ . We thus can write

$$\begin{aligned} I^2 &\lesssim \sum_{N \geq 4} \left( \sum_{4 \leq N_1 \leq 2^{-5}N} \left\| \eta_N(\xi) \eta_{A_\varepsilon}(\xi) |\xi| \chi_{\{|\sigma| \sim \max(N_1 N^2, \varepsilon N^4 N_1)\}} \mathcal{F}_x \left( \Lambda(P_{N_1} u, u_2) \right) \right\|_{X_\varepsilon^{1, -1/2, 1}} \right)^2 \\ &\lesssim \sum_{N \geq 4} \left( \sum_{4 \leq N_1 \leq 2^{-5}N} \|P_{N_1} D_x^{-1/2} u_1\|_{L_{tx}^\infty} \|\chi_{\{|\xi| \sim N\}} \xi \widehat{u_2}\|_{L_{\tau, \xi}^2} \right)^2 \\ &\lesssim \sum_{N \geq 4} \|\chi_{\{|\xi| \sim N\}} \xi \widehat{u_2}\|_{L_{\tau, \xi}^2}^2 \left( \sum_{4 \leq N_1 \leq 2^{-5}N} N_1^{-1/4} \|P_{N_1} D_x^{1/4} u_1\|_{L_t^\infty L_x^2} \right)^2 \\ &\lesssim \|u_1\|_{X^{1/4, 3/4}}^2 \|\partial_x u_2\|_{L_{tx}^2}^2 \\ &\lesssim (\|u_1\|_{X^{0,1}} + \|\partial_x u_1\|_{L_{tx}^2})^2 \|\partial_x u_2\|_{L_{tx}^2}^2. \end{aligned}$$

□

### Proof of the bilinear estimate (2.12)

First, according to (2.14) and Lemma 2.3 and to the support of  $\eta_{A_\varepsilon}$  it suffices to consider

$$I := \left[ \sum_{N \geq 4} N^2 \left( \sum_L L^{-1/2} \left\| \eta_L(\sigma) \eta_N(\xi) \int_{\mathbb{R}^2} \sum_{N_1 \wedge N_2 \geq 8} \widehat{P_{N_1} u_1}(\xi_1, \tau_1) \widehat{P_{N_2} u_2}(\xi_2, \tau_2) d\tau_1 d\xi_1 \right\|_{L_{\tau, \xi}^2(|\xi| \notin J_\varepsilon)} \right)^2 \right]^{1/2},$$

where

$$(2.22) \quad J_\varepsilon = \left[ \frac{15}{16}\sqrt{\frac{3}{5\varepsilon}}, \frac{17}{16}\sqrt{\frac{3}{5\varepsilon}} \right], \quad \tau_2 = \tau - \tau_1 \quad \text{and} \quad \xi_2 = \xi - \xi_1.$$

Now we will decompose the region of integration into different regions and we will check that in most of these regions the strong resonance relation (2.19) holds. By symmetry we can assume that  $N_1 \leq N_2$ . For the remaining it is convenient to introduce the function

$$\Gamma(\xi, \xi_1) := \left| 3 - 5\varepsilon \left( \xi^2 - \xi_1(\xi - \xi_1) \right) \right|$$

which is related to the resonance relation (2.13).

1.  $N_1 < 2^{-10}N_2$ . Then it holds

$$(1 - 2^{-7})\xi^2 \leq \xi^2 - \xi_1(\xi - \xi_1) \leq (1 + 2^{-7})\xi^2$$

and it is easy to check that  $\Gamma(\xi, \xi_1) \geq 2^{-5}$  as soon as  $|\xi| \notin J_\varepsilon$ . According to (2.13) this ensures that (2.19) holds.

2.  $N_1 \geq 2^{-10}N_2$ .

2.1. The subregion  $|\xi| \notin \left[\sqrt{\frac{17}{80\varepsilon}}, \sqrt{\frac{2}{5\varepsilon}}\right]$ . In this region, by (2.5) of Lemma 2.1 and duality, we get

$$\begin{aligned} I &\lesssim \sum_{\min(4, 2^{-10}N_2) < N_1 \leq N_2} \|D_x^{-\frac{1}{4}+} \partial_x^2 (P_{N_1} u_1 P_{N_2} u_2)\|_{L_t^{\frac{4}{3}+} L_x^{1+}} \\ &\lesssim \sum_{\min(4, 2^{-10}N_2) < N_1 \leq N_2} T^{\frac{3}{4}-} N_2^{-\frac{1}{4}+} \|\partial_x P_{N_1} u_1\|_{L_t^\infty L_x^2} \|\partial_x P_{N_2} u_2\|_{L_t^\infty L_x^{2+}} \\ &\lesssim T^{\frac{3}{4}-} \|u_1\|_{L_t^\infty H^1} \|u_2\|_{L_t^\infty H^1}. \end{aligned}$$

2.2. The subregion  $|\xi| \in \left[\sqrt{\frac{17}{80\varepsilon}}, \sqrt{\frac{2}{5\varepsilon}}\right]$ .

2.2.1 The subregion  $|\xi_1| \wedge |\xi_2| \leq \sqrt{\frac{17}{80\varepsilon}}$ . Since both cases can be treated in the same way, we assume  $|\xi_1| \wedge |\xi_2| = |\xi_1|$ . Then, according to (2.5) and the support of  $\eta_{A_\varepsilon}$  and  $\eta_{B_\varepsilon}$ , we get

$$\begin{aligned} I &\lesssim \sum_{\min(4, 2^{-10}N_2) < N_1 \leq N_2} T^{\frac{1}{2}-} \|\partial_x^2 (P_{B_\varepsilon} P_{A_\varepsilon} P_{N_1} u_1 P_{N_2} u_2)\|_{L_{tx}^2} \\ &\lesssim T^{\frac{1}{2}-} \sum_{\min(4, 2^{-10}N_2) < N_1 \leq N_2} \|P_{B_\varepsilon} P_{A_\varepsilon} \partial_x P_{N_1} u_1\|_{L_t^4 L_x^\infty} \|\partial_x P_{N_2} u_2\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}-} \sum_{\min(4, 2^{-10}N_2) < N_1 \leq N_2} N_1^{-1/4} \|P_{A_\varepsilon} P_{N_1} u_1\|_{X^{1,1/2,1}} \|\partial_x P_{N_2} u_2\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}-} \|P_{A_\varepsilon} u_1\|_{X^{1,1/2,1}} \|u_2\|_{L_t^\infty H^1}. \end{aligned}$$

2.2.2 The subregion  $|\xi_1| \wedge |\xi_2| > \sqrt{\frac{17}{80\varepsilon}}$ . In this subregion we claim that (2.19) holds. Indeed, on one hand, if  $\xi_1 \xi_2 \geq 0$  then  $\xi^2 - \xi_1 \xi_2 \leq \xi^2 \leq \frac{2}{5\varepsilon}$  and thus  $\Gamma(\xi, \xi_1) \geq 1$ . On the other hand, if  $\xi_1 \xi_2 \leq 0$  then, since  $|\xi| \geq \sqrt{\frac{17}{80\varepsilon}}$ , we must have  $|\xi_1| \vee |\xi_2| \geq 2\sqrt{\frac{17}{80\varepsilon}}$ . Therefore,  $\xi^2 - \xi_1 \xi_2 \geq 3\frac{17}{80\varepsilon}$  and thus  $\Gamma(\xi, \xi_1) \geq \frac{3}{16}$  which ensures that (2.19) holds and completes the proof of (2.12).

### 3. UNIFORM ESTIMATE CLOSE TO THE STATIONARY POINT OF THE PHASE FUNCTION

As announced in the introduction, close the the stationary point of the phase function we will apply the approach developed by Koch and Tzvetkov in [10]. Note that, in [9], Kenig and Koenig improved this approach by adding the use of the nonlinear local Kato smoothing effect. However, this improvement can not be used here since this smoothing effect is not uniform in  $\varepsilon$  close to the stationary point.

**Proposition 3.1.** *Let  $s \geq 1$  and  $u_\varepsilon \in C([0, T]; H^s(\mathbb{R}))$ ,  $i = 1, 2$ , be a solution to  $(K_\varepsilon)$  with initial data  $\varphi$ . Then it holds*

$$(3.1) \quad \|P_{\mathbb{C}_{A_\varepsilon}} u_\varepsilon\|_{L_T^\infty H_x^s}^2 \lesssim \|P_{\mathbb{C}_{A_\varepsilon}} \varphi\|_{H^s}^2 + (\varepsilon^{1/2} + T^{1/4}) \|u_\varepsilon\|_{Y_{\varepsilon, T}^s}^2 \left( \|u_\varepsilon\|_{Y_{\varepsilon, T}^1} + \|u_\varepsilon\|_{Y_{\varepsilon, T}^1}^2 \right)$$

where  $Y_{\varepsilon, T}^s$  is defined in (2.3) and  $\mathcal{F}_x(P_{\mathbb{C}_{A_\varepsilon}} \varphi) = (1 - \eta_{A_\varepsilon}) \mathcal{F}_x \varphi$ .

First we establish an estimate, uniform in  $\varepsilon$ , on the solution to the associated non homogenous linear problem.

**Lemma 3.1.** *Let  $v \in C([0, T]; H^\infty(\mathbb{R}))$  be a solution of*

$$(3.2) \quad v_t + v_{xxx} + \varepsilon v_{5x} = -F_x \quad .$$

Then

$$(3.3) \quad \|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_T^1 L_x^\infty} \lesssim (\varepsilon^{1/2} + T) \|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_T^\infty L_x^2} + \|P_{\mathbb{C}_{A_\varepsilon}} F\|_{L_T^1 L_x^2} \quad .$$

*Proof.* For  $0 < \varepsilon \ll 1$  fixed, we write a natural splitting

$$[0, T] = \cup I_j$$

of  $[0, T]$  where  $I_j = [a_j, b_j]$  are with disjoint interiors and  $|I_j| \leq \varepsilon^{1/2}$ . Clearly, we can suppose that the number of the intervals  $I_j$  is bounded by  $C(1 + T\varepsilon^{-1/2})$ . Using the Hölder inequality in time, we can write

$$\|v\|_{L_T^1 L_x^\infty} \lesssim \sum_j \|v\|_{L_{I_j}^1 L_x^\infty} \lesssim \varepsilon^{\frac{3}{8}} \sum_j \|v\|_{L_{I_j}^4 L_x^\infty} \quad .$$

Next, we apply the Duhamel formula on each  $I_j$  to obtain

$$P_{\mathbb{C}_{A_\varepsilon}} v(t) = U_\varepsilon(t - a_j) P_{\mathbb{C}_{A_\varepsilon}} v(a_j) - \int_{a_j}^t U_\varepsilon(t - t') P_{\mathbb{C}_{A_\varepsilon}} \partial_x F(t') dt' \quad .$$

Using the uniform in  $\varepsilon$  Strichartz estimate (2.5) and classical  $TT^*$  arguments, it yields

$$\begin{aligned} \|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_{I_j}^4 L_x^\infty} &\lesssim \|D_x^{-1/4} P_{\mathbb{C}_{A_\varepsilon}} v(a_j)\|_{L^2} + \|D_x^{3/4} P_{\mathbb{C}_{A_\varepsilon}} F\|_{L_{I_j}^1 L_x^2} \\ &\lesssim \varepsilon^{1/8} \|P_{\mathbb{C}_{A_\varepsilon}} v(a_j)\|_{L^2} + \varepsilon^{-3/8} \|P_{\mathbb{C}_{A_\varepsilon}} F\|_{L_{I_j}^1 L_x^2} \quad . \end{aligned}$$

Therefore, we get

$$\|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_{I_j}^1 L_x^\infty} \lesssim \varepsilon^{1/2} \|P_{\mathbb{C}_{A_\varepsilon}} v(a_j)\|_{L^2} + \|P_{\mathbb{C}_{A_\varepsilon}} F\|_{L_{I_j}^1 L_x^2}$$

and summing over  $j$ ,

$$\begin{aligned} \|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_T^1 L_{xy}^\infty} &\lesssim \varepsilon^{1/2} \sum_j \|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_T^\infty L_x^2} + \|P_{\mathbb{C}_{A_\varepsilon}} F\|_{L_T^1 L_x^2} \quad . \\ &\lesssim (\varepsilon^{1/2} + T) \|P_{\mathbb{C}_{A_\varepsilon}} v\|_{L_T^\infty L_x^2} + \|P_{\mathbb{C}_{A_\varepsilon}} F\|_{L_T^1 L_x^2} \quad . \end{aligned}$$

□

We now need the following energy estimate

**Lemma 3.2.** *Let  $s \geq 1$ . There exists  $C > 0$  such that all  $0 < \varepsilon \ll 1$  and all  $\varphi \in H^s(\mathbb{R})$ , the solution  $u \in C(0, T; H^s)$  of  $(K_\varepsilon)$  with initial data  $\varphi$  satisfies*

$$(3.4) \quad \|P_{\mathbb{C}_{A_\varepsilon}} u\|_{L_T^\infty H^s}^2 \leq \|P_{\mathbb{C}_{A_\varepsilon}} \varphi\|_{H^s}^2 + C \|P_{B_\varepsilon} u_x\|_{L_T^1 L_x^\infty} \|u\|_{L_T^\infty H^s}^2 \quad .$$

*Proof.* Applying the operator  $P_{\mathbb{C}_{A_\varepsilon}}$  on  $(K_\varepsilon)$  and taking the  $H^s$ -scalar product with  $P_{\mathbb{C}_{A_\varepsilon}} u$  we get

$$\frac{d}{dt} \|P_{\mathbb{C}_{A_\varepsilon}} u(t)\|_{H^s}^2 = \int_{\mathbb{R}} J_x^s P_{\mathbb{C}_{A_\varepsilon}} \partial_x (u^2) J_x^s P_{\mathbb{C}_{A_\varepsilon}} u .$$

Decomposing  $u$  as  $u = P_{B_\varepsilon} u + P_{\mathbb{C}_{B_\varepsilon}} u$  we can rewrite the right-hand side member of the above equality as

$$\int_{\mathbb{R}} J_x^s P_{\mathbb{C}_{A_\varepsilon}} \partial_x (P_{B_\varepsilon} u)^2 J_x^s P_{\mathbb{C}_{A_\varepsilon}} u + \int_{\mathbb{R}} J_x^s P_{\mathbb{C}_{A_\varepsilon}} \partial_x \left( (P_{\mathbb{C}_{B_\varepsilon}} u)^2 + 2P_{B_\varepsilon} u P_{\mathbb{C}_{B_\varepsilon}} u \right) J_x^s P_{\mathbb{C}_{A_\varepsilon}} u := I_1 + I_2 .$$

In the sequel we will need the following variant of the Kato-Ponce commutator estimate ([7]):

$$(3.5) \quad \left\| \left[ J_x^s P_{\mathbb{C}_{A_\varepsilon}}, f \right] g \right\|_{L_x^2} \lesssim \|f\|_{L_x^\infty} \|g\|_{H_x^{s-1}} + \|f\|_{H_x^s} \|g\|_{L_x^\infty} .$$

Integrating by parts and applying the above commutator estimate we easily estimate the first term by

$$\begin{aligned} I_1 &= 2 \int_{\mathbb{R}} P_{B_\varepsilon} u \partial_x \left( J_x^s P_{\mathbb{C}_{A_\varepsilon}} P_{B_\varepsilon} u \right) J_x^s P_{\mathbb{C}_{A_\varepsilon}} u + 2 \int_{\mathbb{R}} \left[ J_x^s P_{\mathbb{C}_{A_\varepsilon}}, P_{B_\varepsilon} u \right] P_{B_\varepsilon} u J_x^s P_{\mathbb{C}_{A_\varepsilon}} u \\ &\lesssim \|P_{B_\varepsilon} u\|_{L_x^\infty} \|u\|_{H^s}^2 , \end{aligned}$$

where, in the last step, we use that according to the support localization of  $\eta$ ,

$$(3.6) \quad P_{B_\varepsilon} P_{\mathbb{C}_{A_\varepsilon}} = P_{\mathbb{C}_{A_\varepsilon}} .$$

For the second term, we notice that by the frequency projections, all the functions in the integral are supported in frequencies of order  $1/\sqrt{\varepsilon}$ . Therefore, using Bernstein inequalities we get

$$\begin{aligned} I_2 &\lesssim \varepsilon^{-s-1/2} \left\| P_{\mathbb{C}_{A_\varepsilon}} \left( (P_{\mathbb{C}_{B_\varepsilon}} u)^2 + 2\mathcal{F}_x^{-1} \left( \chi_{\{|\xi| \sim \varepsilon^{-\frac{1}{2}}\}} \mathcal{F}(P_{B_\varepsilon} u) \right) P_{\mathbb{C}_{B_\varepsilon}} u \right) \right\|_{L_x^1} \|P_{\mathbb{C}_{A_\varepsilon}} u\|_{L_x^\infty} \\ &\lesssim \|P_{\mathbb{C}_{A_\varepsilon}} u\|_{L_x^\infty} \|u\|_{H^s}^2 . \end{aligned}$$

(3.4) then follows by integration in time, using again (3.6).  $\square$

**Proof of Proposition 3.1** Applying (3.3) to  $u_x$  with  $u$  solving  $(K_\varepsilon)$  we get

$$\begin{aligned} \|P_{\mathbb{C}_{A_\varepsilon}} u_x\|_{L_T^1 L_x^\infty} &\lesssim (\varepsilon^{1/2} + T) \|P_{\mathbb{C}_{A_\varepsilon}} u_x\|_{L_T^\infty L_x^2} + \|P_{\mathbb{C}_{A_\varepsilon}} \partial_x (u^2)\|_{L_T^1 L_x^2} \\ (3.7) \quad &\lesssim (\varepsilon^{1/2} + T) \|u\|_{L_T^\infty H_x^1} + T \|u\|_{L_T^\infty H_x^1}^2 . \end{aligned}$$

Therefore, gathering (3.4), (3.7) and (2.5) we obtain

$$\begin{aligned} \|P_{\mathbb{C}_{A_\varepsilon}} u\|_{L_T^\infty H_x^s}^2 &\lesssim \|P_{\mathbb{C}_{A_\varepsilon}} u_0\|_{H^s}^2 + C \|u\|_{L_T^\infty H_x^s}^2 \left( T^{1/4} \|P_{B_\varepsilon} P_{A_\varepsilon} u_x\|_{L_T^4 L_x^\infty} + \|P_{\mathbb{C}_{A_\varepsilon}} u_x\|_{L_T^1 L_x^\infty} \right) \\ &\lesssim \|P_{\mathbb{C}_{A_\varepsilon}} u_0\|_{H^s}^2 + C (\varepsilon^{1/2} + T^{1/4}) \|u\|_{L_T^\infty H_x^s}^2 \left( \|u\|_{Y_{\varepsilon,T}^1} + \|u\|_{Y_{\varepsilon,T}^1}^2 \right) , \end{aligned}$$

which completes the proof of (3.1).  $\square$

## 4. PROOF OF THEOREM 1.2

**4.1. Uniform bound on the solutions.** Let  $u \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}))$  be a solution of  $(K_\varepsilon)$ . Combining Propositions 2.1 and 3.1 we infer that for any  $s \geq 1$  and  $T \in ]0, 1[$ ,

$$\|u\|_{Y_{\varepsilon,T}^s}^2 \leq C \|\varphi\|_{H^s}^2 + C(\sqrt{\varepsilon} + T^{1/4}) \|u\|_{Y_{\varepsilon,T}^s}^2 \|u\|_{Y_{\varepsilon,T}^1} \left(1 + \|u\|_{Y_{\varepsilon,T}^1}^3\right),$$

for some constant  $C > 0$ . Since  $u$  is smooth,  $T \mapsto \|u\|_{Y_{\varepsilon,T}^s}$  is continuous and  $\limsup_{T \searrow 0} \|u\|_{Y_{\varepsilon,T}^s} \lesssim \|\varphi\|_{H^s}$ . Therefore a classical continuity argument ensures that for any  $\delta > 0$  there exists  $\alpha > 0$  such that

$$(4.1) \quad C(\sqrt{\varepsilon} + T^{1/4}) \|u\|_{Y_{\varepsilon,T}^s}^2 \|u\|_{Y_{\varepsilon,T}^1} \left(1 + \|u\|_{Y_{\varepsilon,T}^1}^3\right) \leq \delta$$

and  $\|u\|_{Y_{\varepsilon,T}^s} \lesssim \|\varphi\|_{H^s}$  provided

$$(4.2) \quad (\sqrt{\varepsilon} + T^{1/4}) \leq \alpha(\|\varphi\|_{H^1} + \|\varphi\|_{H^1}^4)^{-1}.$$

By continuity with respect to initial data (for any fixed  $\varepsilon > 0$ ) it follows that for any fixed initial data  $\varphi \in H^s(\mathbb{R})$ ,  $s \geq 1$ , the emanating solution  $u \in C(\mathbb{R}; H^s(\mathbb{R}))$  of  $(K_\varepsilon)$ , with

$$(4.3) \quad 0 < \varepsilon \leq \varepsilon_0(\|\varphi\|_{H^1}) := \frac{\alpha^2}{4}(\|\varphi\|_{H^1} + \|\varphi\|_{H^1}^4)^{-2},$$

satisfies

$$(4.4) \quad \|u\|_{Y_{\varepsilon,T}^s} \lesssim \|\varphi\|_{H^s},$$

with  $T = T(\|\varphi\|_{H^1}) \sim (\|\varphi\|_{H^1} + \|\varphi\|_{H^1}^4)^{-4}$ .

Finally, the result for  $\varepsilon \in [\varepsilon_0(\|\varphi\|_{H^1}), 1]$  follows from a dilation argument. Indeed, it is easy to check that  $u$  is a solution of  $(K_\varepsilon)$  with initial data  $\varphi$  if and only if  $u_\lambda = u_\lambda(t, x) = \lambda^{-2}u(\lambda^{-3}t, \lambda^{-1}x)$  is a solution of  $(K_{\lambda^2\varepsilon})$  with initial data  $\varphi_\lambda = \lambda^{-2}\varphi(\lambda^{-1}x)$ . Hence, taking  $\lambda = \varepsilon^{-1/2} \geq 1$  we observe that  $u_\lambda$  satisfies  $(K_1)$ . By classical well-posedness result for  $(K_1)$  (see for instance [6]), there exists a non increasing function  $R : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  such that

$$\|u_\lambda\|_{L_T^\infty H^s} \lesssim \|\varphi_\lambda\|_{H^s} \text{ with } T' = R(\|\varphi_\lambda\|_{H^1}).$$

Coming back to  $u$ , noticing that  $\|\varphi_\lambda\|_{H^1} \lesssim \lambda^{-3/2}\|\varphi\|_{H^1}$  and that  $1 \leq \lambda = \varepsilon^{-1/2} \lesssim (\|\varphi\|_{H^1} + \|\varphi\|_{H^1}^2)$  we deduce that

$$\|u\|_{L_T^\infty H^s} \lesssim \|\varphi\|_{H^s} \text{ with } T = T(\|\varphi\|_{H^1}),$$

which completes the proof of (1.4).

**4.2. Proof the equi-continuity result.** Now to prove the equi-continuity result we will make use of Bona-Smith argument [2]. To simplify the expository we will only consider the most difficult case that is the case  $s = 1$ . We thus want to prove that, be given a sequence  $\{\varphi_k\} \subset H^1(\mathbb{R})$  converging towards  $\varphi$  in  $H^1(\mathbb{R})$ , the emanating solutions  $u_{\varepsilon,k} := S_{K_\varepsilon}(\varphi_k)$  satisfy

$$(4.5) \quad \lim_{k \rightarrow \infty} \sup_{0 < \varepsilon < 1} \|u_{\varepsilon,k} - u_\varepsilon\|_{L_T^\infty H^1} = 0,$$

where  $u_\varepsilon := S_{K_\varepsilon}(\varphi)$  and  $T = T(\|\varphi\|_{H^1})$ . We first notice that we can restrict ourself to consider  $\varepsilon$  satisfying (4.3) since the same dilation argument as above yields directly the result otherwise.

The first step consists in repeating the arguments of Sections 2 & 3 to get a  $L^2$ -Lipschitz bound, uniform in  $\varepsilon$ , for  $H^1$ -solution. This is the aim of the following proposition which proof is postponed in the appendix.

**Proposition 4.1.** *Let  $0 < \varepsilon < 1$ ,  $T > 0$  and  $v \in Y_{\varepsilon, T}^1$  satisfying*

$$(4.6) \quad (\sqrt{\varepsilon} + T^{1/4}) \left( \|v\|_{Y_{\varepsilon, T}^1} + \|v\|_{Y_{\varepsilon, T}^1}^4 \right) \ll 1$$

and

$$(4.7) \quad \|P_{B_\varepsilon} \partial_x v\|_{L_T^1 L_x^\infty} \lesssim (\sqrt{\varepsilon} + T^{1/4}) \left( \|v\|_{Y_{\varepsilon, T}^1} + \|v\|_{Y_{\varepsilon, T}^1}^2 \right).$$

Then any solution  $w \in C([0, T]; H^1(\mathbb{R}))$  to

$$(4.8) \quad \partial_t w + \partial_x^3 w + \varepsilon \partial_x^5 w + \frac{1}{2} \partial_x(wv) = 0$$

satisfies

$$(4.9) \quad \|w\|_{L_T^\infty L_x^2} \lesssim \|w(0)\|_{L_x^2}$$

where the implicit constant is independent of  $\varepsilon$ .

Now, for any  $\varphi \in H^1(\mathbb{R})$  and any dyadic integer  $N$  we set  $\varphi^N := P_{\leq N} \varphi$ . By straightforward calculations in Fourier space, for any  $\varphi \in H^1(\mathbb{R})$ , any  $N \geq 1$  and any  $r \geq 0$ ,

$$(4.10) \quad \|\varphi^N\|_{H_x^{1+r}} \lesssim N^r \|\varphi\|_{H_x^1} \quad \text{and} \quad \|\varphi^N - \varphi\|_{H_x^{1-r}} \lesssim o(N^{-r}) \|\varphi\|_{H_x^1}.$$

Setting  $u_\varepsilon^N := S_{K_\varepsilon}(\varphi^N)$  and  $u_{\varepsilon, k}^N := S_{K_\varepsilon}(\varphi_k^N)$ , (4.1) ensures that there exists  $T_0 = T_0(\|\varphi\|_{H^1}) \in ]0, 1[$  such that for  $k$  large enough and  $z := u_\varepsilon$ ,  $u_\varepsilon^N$ ,  $u_{\varepsilon, k}$  or  $u_{\varepsilon, k}^N$ ,

$$(4.11) \quad (\sqrt{\varepsilon} + T_0^{1/4}) \left( \|z\|_{Y_{\varepsilon, T_0}^1} + \|z\|_{Y_{\varepsilon, T_0}^1}^4 \right) \ll 1$$

and, according to (4.4), (2.5) and (3.7),

$$(4.12) \quad \|z\|_{Y_{\varepsilon, T_0}^1} \leq 2\|\varphi\|_{H^1} \quad \text{and} \quad \|P_{B_\varepsilon} \partial_x z\|_{L_{T_0}^1 L_x^\infty} \lesssim (\sqrt{\varepsilon} + T_0^{1/4}) \left( \|z\|_{Y_{\varepsilon, T_0}^1} + \|z\|_{Y_{\varepsilon, T_0}^1}^2 \right).$$

Moreover,

$$(4.13) \quad \|u_\varepsilon^N\|_{Y_{\varepsilon, T_0}^s} + \|u_{\varepsilon, k}^N\|_{Y_{\varepsilon, T_0}^s} \lesssim \|\varphi^N\|_{H^s} \lesssim N^{s-1} \|\varphi\|_{H^1}$$

provided  $s \geq 1$ . By the triangle inequality, it holds

$$(4.14) \quad \|u_\varepsilon - u_{\varepsilon, k}\|_{L_T^\infty H_x^1} \leq \|u_\varepsilon - u_\varepsilon^N\|_{L_T^\infty H_x^1} + \|u_\varepsilon^N - u_{\varepsilon, k}^N\|_{L_T^\infty H_x^1} + \|u_{\varepsilon, k}^N - u_{\varepsilon, k}\|_{L_T^\infty H_x^1}.$$

We start by estimating the first term of the right-hand side of (4.14). Setting  $w_\varepsilon := u_\varepsilon - u_\varepsilon^N$ , we observe that  $w_\varepsilon$  satisfies

$$(4.15) \quad \partial_t w_\varepsilon + \partial_x^3 w_\varepsilon + \varepsilon \partial_x^5 w_\varepsilon + \frac{1}{2} \partial_x(w_\varepsilon(u_\varepsilon^N + u_\varepsilon)) = 0.$$

Therefore, combining Proposition 4.1, (4.11)-(4.12) and (4.10) we get that

$$(4.16) \quad \|w_\varepsilon\|_{L_{T_0}^\infty L_x^2} \lesssim o(N^{-1}).$$

According to (2.2) we also have

$$(4.17) \quad \|P_{A_\varepsilon} w_\varepsilon\|_{X_{\varepsilon, T_0}^{1, 1/2, 1}} \leq C \|\varphi - \varphi^N\|_{H_x^1} + \frac{1}{2} \|w_\varepsilon\|_{Y_{\varepsilon, T_0}^1}.$$

Now to estimate  $P_{\mathbb{C}A_\varepsilon}$  we rewrite the equation satisfying by  $w_\varepsilon$  in the following less symmetric way :

$$\partial_t w_\varepsilon + \partial_x^3 w_\varepsilon + \varepsilon \partial_x^5 w_\varepsilon = -\frac{1}{2} \partial_x (w_\varepsilon^2) - \partial_x (u_\varepsilon^N w_\varepsilon) .$$

Applying the operator  $P_{\mathbb{C}A_\varepsilon}$  on the above equation and taking the  $H^1$ -scalar product with  $P_{\mathbb{C}A_\varepsilon} w_\varepsilon$  we get

$$\begin{aligned} \frac{d}{dt} \|P_{\mathbb{C}A_\varepsilon} w_\varepsilon(t)\|_{H_x^1}^2 &= \int_{\mathbb{R}} J_x^1 P_{\mathbb{C}A_\varepsilon} \partial_x (w_\varepsilon^2) J_x^1 P_{\mathbb{C}A_\varepsilon} w_\varepsilon + 2 \int_{\mathbb{R}} J_x^1 P_{\mathbb{C}A_\varepsilon} (u_\varepsilon^N \partial_x w_\varepsilon) J_x^1 P_{\mathbb{C}A_\varepsilon} w_\varepsilon \\ (4.18) \quad &+ 2 \int_{\mathbb{R}} J_x^1 P_{\mathbb{C}A_\varepsilon} (w_\varepsilon \partial_x u_\varepsilon^N) J_x^1 P_{\mathbb{C}A_\varepsilon} w_\varepsilon . \end{aligned}$$

The contribution of the first term of the above right-hand side can be estimated in exactly the same way as in the proof of Lemma 3.2 by  $\|P_{B_\varepsilon} \partial_x w_\varepsilon\|_{L_x^\infty} \|w_\varepsilon\|_{H_x^1}^2$ . The second term can be estimated also in the same way by

$$(\|P_{B_\varepsilon} \partial_x u_\varepsilon^N\|_{L_x^\infty} \|w_\varepsilon\|_{H_x^1} + \|u_\varepsilon^N\|_{H_x^1} \|P_{B_\varepsilon} \partial_x w_\varepsilon\|_{L_x^\infty}) \|w_\varepsilon\|_{H_x^1} .$$

The difficulty comes from the third term. To estimate its contribution we first decompose  $w_\varepsilon$  and  $u_\varepsilon^N$  to rewrite it as

$$\begin{aligned} 2 \int_{\mathbb{R}} J_x^1 P_{\mathbb{C}A_\varepsilon} \left( P_{\mathbb{C}B_\varepsilon} w_\varepsilon P_{\mathbb{C}B_\varepsilon} \partial_x u_\varepsilon^N + P_{\mathbb{C}B_\varepsilon} w_\varepsilon P_{B_\varepsilon} \partial_x u_\varepsilon^N + P_{B_\varepsilon} w_\varepsilon P_{\mathbb{C}B_\varepsilon} \partial_x u_\varepsilon^N \right) J_x^1 P_{\mathbb{C}A_\varepsilon} w_\varepsilon \\ + 2 \int_{\mathbb{R}} J_x^1 P_{\mathbb{C}A_\varepsilon} (P_{B_\varepsilon} w_\varepsilon P_{B_\varepsilon} \partial_x u_\varepsilon^N) J_x^1 P_{\mathbb{C}A_\varepsilon} w_\varepsilon = I_1 + I_2 \end{aligned}$$

According to the frequency projections, in the same way as proof of Lemma 3.2, all the functions in  $I_1$  are supported in frequencies of order  $\varepsilon^{-1/2}$ , which leads to

$$I_1 \lesssim \|P_{\mathbb{C}A_\varepsilon} \partial_x w_\varepsilon\|_{L_x^\infty} \|w_\varepsilon\|_{H_x^1} \|u_\varepsilon^N\|_{H_x^1} .$$

Finally we control the contribution of  $I_2$  by

$$I_2 \lesssim \|P_{\mathbb{C}A_\varepsilon} \partial_x w_\varepsilon\|_{L_x^\infty} \left( \|w_\varepsilon\|_{H^1} \|u_\varepsilon^N\|_{H_x^1} + \|w_\varepsilon\|_{L^2} \|u_\varepsilon^N\|_{H^2} \right)$$

Note that the difficulty to control  $I_2$  comes from the fact that we can not avoid to put a  $H^2$ -norm on  $u_\varepsilon^N$ . But the idea of Bona-Smith is to compensate the growth with  $N$  of this  $H^2$ -norm by the decay with  $N$  of the  $L^2$ -norm of  $w_\varepsilon$ . Actually, integrating (4.18) in time, with the above estimates together with (4.16) and (4.12)-(4.13) in hand, we get

$$\begin{aligned} \|P_{\mathbb{C}A_\varepsilon} w_\varepsilon\|_{L_{T_0}^\infty H_x^1}^2 &\lesssim \|P_{\mathbb{C}A_\varepsilon} (\varphi - \varphi^N)\|_{H_x^1}^2 + (\|P_{B_\varepsilon} \partial_x w_\varepsilon\|_{L_{T_0}^1 L_x^\infty} + \|P_{B_\varepsilon} \partial_x u_\varepsilon^N\|_{L_{T_0}^1 L_x^\infty}) \|w_\varepsilon\|_{L_{T_0}^\infty H_x^1}^2 \\ &\quad + \|P_{B_\varepsilon} \partial_x w_\varepsilon\|_{L_{T_0}^1 L_x^\infty} (\|w_\varepsilon\|_{L_{T_0}^\infty H_x^1} \|u_\varepsilon^N\|_{L_{T_0}^\infty H_x^1} + \|w_\varepsilon\|_{L_{T_0}^\infty L_x^2} \|u_\varepsilon^N\|_{L_{T_0}^\infty H_x^2}) \\ &\leq \gamma_1(N) + \frac{1}{8} \|w_\varepsilon\|_{L_{T_0}^\infty H_x^1}^2 + 2 \|\varphi\|_{H_x^1} \|P_{B_\varepsilon} \partial_x w_\varepsilon\|_{L_{T_0}^1 L_x^\infty} \|w_\varepsilon\|_{L_{T_0}^\infty H_x^1} \\ (4.19) \quad &+ o(N^{-1})N \end{aligned}$$

where  $\gamma_1(N) \rightarrow 0$  as  $N \rightarrow \infty$ . On the other hand, applying Lemma 3.1 on the  $x$ -derivative of (4.15) we get

$$\begin{aligned} (4.20) \quad \|P_{\mathbb{C}A_\varepsilon} \partial_x w_\varepsilon\|_{L_{T_0}^1 L_x^\infty} &\lesssim (\varepsilon^{1/2} + T_0) \|w_\varepsilon\|_{L_{T_0}^\infty H_x^1} + T_0 \|w_\varepsilon\|_{L_{T_0}^\infty H_x^1} (\|u_\varepsilon\|_{L_{T_0}^\infty H^1} + \|u_\varepsilon^N\|_{L_{T_0}^\infty H^1}) \end{aligned}$$



Therefore, gathering (4.19), (4.20) and (4.17) with (4.11) in hand, we obtain

$$\|w_\varepsilon\|_{Y_{T_0}^1}^2 \leq \gamma_2(N) + \frac{1}{2}\|w_\varepsilon\|_{Y_{T_0}^1}^2$$

with  $\gamma_2(N) \rightarrow 0$  as  $N \rightarrow \infty$ . This ensures that

$$\|u_\varepsilon - u_\varepsilon^N\|_{L_{T_0}^\infty H_x^1} \leq 2\gamma_2(N).$$

To estimate the contribution of the third term of the right-hand side of (4.14) we proceed exactly in the same way as for the first one, by replacing  $u_\varepsilon$  by  $u_{\varepsilon,k}$  and  $u_\varepsilon^N$  by  $u_{\varepsilon,k}^N$ . We then obtain

$$\|u_{\varepsilon,k} - u_{\varepsilon,k}^N\|_{L_{T_0}^\infty H_x^1} \leq \gamma_3(N).$$

with  $\gamma_3(N) \rightarrow 0$  as  $N \rightarrow \infty$ . Finally, the contribution of the second term of the right-hand side of (4.14) is also obtain in the same way by replacing  $u_\varepsilon$  by  $u_{\varepsilon,k}^N$  (actually, contrary to the preceding contributions, here both terms  $u_{\varepsilon,k}^N$  and  $u_\varepsilon^N$  can play a symmetric role). However, for this term, Proposition 4.1 only ensures that

$$\|u_\varepsilon^N - u_{\varepsilon,k}^N\|_{L_{T_0}^\infty L_x^2} \lesssim \|\varphi - \varphi_k\|_{L_x^2}.$$

Therefore, setting  $w_\varepsilon = u_\varepsilon^N - u_{\varepsilon,k}^N$ , one has to replace  $o(N^{-1})N$  by  $\|\varphi - \varphi_k\|_{L_x^2}N$  in the right-hand side member of (4.19) when estimating  $\|P_{\mathbb{C}A_\varepsilon}(u_\varepsilon^N - u_{\varepsilon,k}^N)\|_{L_{T_0}^\infty H_x^1}$ . We thus obtain

$$\|u_\varepsilon^N - u_{\varepsilon,k}^N\|_{L_{T_0}^\infty H_x^1} \lesssim \|\varphi - \varphi_k\|_{H_x^1} + N\|\varphi - \varphi_k\|_{L_x^2}.$$

Gathering the above estimates, (4.14) leads to

$$\lim_{k \rightarrow +\infty} \sup_{0 < \varepsilon < \varepsilon_0(\|\varphi\|_{H^1})} \|u_\varepsilon - u_{\varepsilon,k}\|_{L_{T_0}^\infty H_x^1} = 0$$

which completes the proof of Theorem 1.2.

**4.3. Proof of Theorem 1.1.** We follow general arguments (see for instance [5]). Let us denote by  $S_{K_\varepsilon}$  and  $S_{KdV}$  the nonlinear group associated with respectively  $(K_\varepsilon)$  and KdV. Let  $\varphi \in H_x^s(\mathbb{R})$ ,  $s \geq 1$  and let  $T = T(\|\varphi\|_{H_x^1}) > 0$  be given by Theorem 1.1. For any  $N > 0$  we can rewrite  $S_{K_\varepsilon}(\varphi) - S_{KdV}(\varphi)$  as

$$\begin{aligned} S_{K_\varepsilon}(\varphi) - S_{KdV}(\varphi) &= \left( S_{K_\varepsilon}(\varphi) - S_{K_\varepsilon}(P_{\leq N}\varphi) \right) + \left( S_{K_\varepsilon}(P_{\leq N}\varphi) - S_{KdV}(P_{\leq N}\varphi) \right) \\ &\quad + \left( S_{KdV}(P_{\leq N}\varphi) - S_{KdV}(\varphi) \right) = I_{\varepsilon,N} + J_{\varepsilon,N} + K_N. \end{aligned}$$

By continuity with respect to initial data in  $H^s(\mathbb{R})$  of the solution map associated with the KdV equation, we have  $\lim_{N \rightarrow \infty} \|K_N\|_{L^\infty(0,T;H_x^s)} = 0$ . On the other hand, (1.5) ensures that

$$\lim_{N \rightarrow \infty} \sup_{\varepsilon \in ]0,1[} \|I_{\varepsilon,N}\|_{L^\infty(0,T;H_x^s)} = 0.$$

It thus remains to check that for any fixed  $N > 0$ ,  $\lim_{\varepsilon \rightarrow 0} \|J_{\varepsilon,N}\|_{L^\infty(0,T;H_x^s)} = 0$ . Since  $P_{\leq N}\varphi \in H^\infty(\mathbb{R})$ , it is worth noticing that  $S_{K_\varepsilon}(P_{\leq N}\varphi)$  and  $S_{KdV}(P_{\leq N}\varphi)$  belong to  $C^\infty(\mathbb{R}; H^\infty(\mathbb{R}))$ . Moreover, according to Theorem 1.2 and the well-posedness theory of the KdV equation (see for instance [1]), for all  $\theta \in \mathbb{R}$  and  $\varepsilon \in ]0,1[$ ,

$$\|S_{K_\varepsilon}(P_{\leq N}\varphi)\|_{L_T^\infty H_x^\theta} + \|S_{KdV}(P_{\leq N}\varphi)\|_{L_T^\infty H_x^\theta} \leq C(N, \theta, \|\varphi\|_{L_x^2}).$$

Now, setting  $v_\varepsilon := S_{K_\varepsilon}(P_{\leq N}\varphi)$  and  $v := S_{KdV}(P_{\leq N}\varphi)$ , we observe that  $w_\varepsilon := v_\varepsilon - v$  satisfies

$$\partial_t w_\varepsilon + \partial_x^3 w_\varepsilon + \varepsilon \partial_x^5 w_\varepsilon = -\frac{1}{2} \partial_x \left( w_\varepsilon (v + v_\varepsilon) \right) - \varepsilon v_{5x}$$

with initial data  $w_\varepsilon(0) = 0$ . Taking the  $H^s$ -scalar product of this last equation with  $w_\varepsilon$  and integrating by parts we get

$$\frac{d}{dt} \|w_\varepsilon\|_{H_x^s}^2 \lesssim \left(1 + \|\partial_x(v + v_\varepsilon)\|_{L_x^\infty}\right) \|w_\varepsilon\|_{H_x^s}^2 + \|[J^s, (v + v_\varepsilon)] \partial_x w_\varepsilon\|_{L^2} \|w_\varepsilon\|_{H_x^s} + \varepsilon^2 \|v_{5x}\|_{H_x^s}^2.$$

Making use of the following commutator estimate (see for instance [12]), that holds for  $s > 1/2$ ,

$$(4.21) \quad \|[J_x^s, f]g\|_{L_x^2} \lesssim \|f_x\|_{H_x^s} \|g\|_{H_x^{s-1}},$$

we easily get

$$\frac{d}{dt} \|w_\varepsilon(t)\|_{H_x^s}^2 \lesssim C(N, s+1, \|\varphi\|_{L_x^2}) \|w_\varepsilon(t)\|_{H_x^s}^2 + \varepsilon^2 C(N, 5+s, \|\varphi\|_{L_x^2})^2.$$

Integrating this differential inequality on  $[0, T]$ , this ensures that  $\lim_{\varepsilon \rightarrow 0} \|w_\varepsilon\|_{L^\infty(0, T; H^s)} = 0$  and completes the proof of Theorem 1.1 with  $T = T(\|\varphi\|_{H^1})$ . Finally, recalling that the energy conservation of the KdV equation ensures that for any  $\varphi \in H^1(\mathbb{R})$  it holds,

$$\sup_{t \in \mathbb{R}} \|S_{KdV}(\varphi)(t)\|_{H_x^1} \lesssim \|\varphi\|_{H_x^1} + \|\varphi\|_{L_x^2}^5,$$

we obtain the same convergence result on any time interval  $[0, T_0]$  with  $T_0 > T(\|\varphi\|_{H_x^1})$  by reiterating the convergence result about  $T_0/T(\|\varphi\|_{H_x^1} + \|\varphi\|_{L_x^2}^5)$  times.

## 5. APPENDIX: PROOF OF PROPOSITION 4.1

We follow very closely Sections 2 and 3. The first step consists in establishing the following estimate on  $P_{A_\varepsilon} w$ .

**Proposition 5.1.** *Let  $0 < T < 1$  and  $w \in C([0, T]; H^1(\mathbb{R}))$  be a solution to (4.8) with  $0 < \varepsilon \ll 1$  and initial data  $\varphi$ . Then it holds*

$$(5.1) \quad \|P_{A_\varepsilon} w\|_{X_{\varepsilon, T}^{0, 1/2, 1}} \lesssim \|\varphi\|_{L^2} + T^{\frac{1}{4}-} \|v\|_{Y_{\varepsilon, T}^1} \|w\|_{Y_{\varepsilon, T}^0} (1 + \|v\|_{Y_{\varepsilon, T}^1})$$

*Proof.* We proceed as in Section 2. First we observe that we have trivially

$$(5.2) \quad \|P_{\leq 8} \partial_x(vw)\|_{X_{\varepsilon}^{0, -1/2, 1}} \lesssim \|vw\|_{L_{tx}^2} \lesssim T^{1/2} \|v\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty L_x^2}.$$

and

$$(5.3) \quad \begin{aligned} \|P_{A_\varepsilon} \partial_x(vP_{\leq 8} w)\|_{X_{\varepsilon}^{0, -1/2, 1}} &\lesssim \|P_{A_\varepsilon} \partial_x(vP_{\leq 8} w)\|_{L_{tx}^2} \\ &\lesssim T^{1/2} \left( \|v_x\|_{L_t^\infty L_x^2} \|P_{\leq 8} w\|_{L_{tx}^\infty} + \|v\|_{L_{tx}^\infty} \|w\|_{L_t^\infty L_x^2} \right) \\ &\lesssim T^{1/2} \|v\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty L_x^2}. \end{aligned}$$

Now to control  $\|P_{A_\varepsilon} \partial_x(wP_{\leq 8} v)\|_{X_{\varepsilon}^{0, -1/2, 1}}$  we notice that in the same way as in (5.3) we have

$$\|P_{A_\varepsilon} \partial_x(P_{\leq 16} w P_{\leq 8} v)\|_{X_{\varepsilon}^{0, -1/2, 1}} \lesssim T^{1/2} \|v\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty L_x^2}.$$

On the other hand, according to the frequency projections and Lemma 2.3, the contribution of  $P_{\geq 16}w$  can be estimated by

$$\begin{aligned}
 \|\partial_x P_{A_\varepsilon}(P_{\leq 8}vP_{\geq 16}w)\|_{X_\varepsilon^{0,-1/2,1}} &= \|\partial_x P_{A_\varepsilon}\Lambda(P_{\leq 8}v, P_{\geq 16}w)\|_{X_\varepsilon^{0,-1/2,1}} \\
 &\lesssim \left\| \partial_x P_{A_\varepsilon}\Lambda\left(P_{\leq 8}\mathcal{F}_{xt}^{-1}(|\widehat{v}|), \partial_x^{-1}P_{\geq 16}\mathcal{F}_{xt}^{-1}(|\widehat{w}|)\right) \right\|_{X_\varepsilon^{1,-1/2,1}} \\
 (5.4) \quad &\lesssim \|v\|_{X_\varepsilon^{0,1}} \left( T^{1/4}(\|P_{A_\varepsilon}w\|_{X_\varepsilon^{0,1/2,1}} + \|w\|_{X_\varepsilon^{-1,1}}) + \|w\|_{L_{tx}^2} \right).
 \end{aligned}$$

To continue we need the following variant of Lemma 2.4.

**Lemma 5.1.** *Let  $v$  and  $w$  be two smooth functions supported in time in  $] - T, T[$  with  $0 < T \leq 1$ . Then, in the region where the strong resonance relation (2.19) holds, we have*

$$\begin{aligned}
 \|\partial_x P_{A_\varepsilon}P_{\geq 8}(P_{\geq 8}vP_{\geq 8}w)\|_{X_\varepsilon^{0,-1/2,1}} &\lesssim T^{\frac{1}{4}-}\|v\|_{X_\varepsilon^{0,1}}\|w\|_{X_\varepsilon^{-1,1}} + \|v_x\|_{L_{tx}^2}(\|w\|_{X_\varepsilon^{-1,1}} + \|w\|_{L_{tx}^2}) \\
 (5.5) \quad &+ \|w\|_{L_{tx}^2}(\|v\|_{X_\varepsilon^{0,1}} + \|v_x\|_{L_{tx}^2}).
 \end{aligned}$$

*Proof.* We notice that the norms in the right-hand side of (2.4) only see the size of the modulus of the Fourier transforms. We can thus assume that all our functions have non-negative Fourier transforms. We set  $I := \|\partial_x P_{A_\varepsilon}P_{\geq 8}(P_{\geq 8}vP_{\geq 8}w)\|_{X_\varepsilon^{0,-1/2,1}}$  and separate different subregions .

- $|\sigma_2| \geq 2^{-5}|\xi\xi_1(\xi - \xi_1)|$ . Then direct calculations give

$$\begin{aligned}
 I &\lesssim T^{\frac{1}{2}-}\|w\|_{X_\varepsilon^{-1,1}}\|D_x^{-1}P_{\geq 8}v\|_{L_{tx}^\infty} \\
 &\lesssim T^{\frac{1}{2}-}\|v\|_{X_\varepsilon^{0,1}}\|w\|_{X_\varepsilon^{-1,1}}.
 \end{aligned}$$

- $|\sigma_1| \geq 2^{-5}|\xi\xi_1(\xi - \xi_1)|$ . Then , by (2.5) of Lemma 2.1 and duality, we get

$$\begin{aligned}
 I &\lesssim \left\| P_{A_\varepsilon}P_{\geq 8}\left(P_{\geq 8}D_x^{-1}\mathcal{F}_{tx}^{-1}(\langle\sigma_1\rangle\widehat{u_1})P_{\geq 8}D_x^{-1}w\right) \right\|_{L_t^{\frac{4}{3}+}L_x^{1+}} \\
 &\lesssim T^{\frac{1}{4}-}\|v\|_{X_\varepsilon^{0,1}}\|P_{\geq 8}D_x^{-1}w\|_{L_t^\infty L_x^{2+}} \\
 &\lesssim T^{\frac{1}{4}-}\|v\|_{X_\varepsilon^{0,1}}\|w\|_{X_\varepsilon^{-3/4,3/4}} \\
 &\lesssim T^{\frac{1}{4}-}\|v\|_{X_\varepsilon^{0,1}}(\|w\|_{X_\varepsilon^{-1,1}} + \|w\|_{L_{tx}^2}).
 \end{aligned}$$

- $|\sigma| \geq 2^{-5}|\xi\xi_1(\xi - \xi_1)|$  and  $\max(|\sigma_1|, |\sigma_2|) \leq 2^{-5}|\xi\xi_1(\xi - \xi_1)|$ .

Then we separate two subregions.

1.  $|\xi_1| \wedge |\xi_2| \geq 2^{-7}|\xi|$ . Then  $|\xi_1| \sim |\xi_2| \gtrsim |\xi|$  and taking  $\delta > 0$  close enough to 0 we get

$$\begin{aligned}
 I &\lesssim \|\partial_x P_{A_\varepsilon}P_{\geq 8}(P_{\geq 8}vP_{\geq 8}w)\|_{X_\varepsilon^{0,-1/2+\delta}} \\
 &\lesssim \left\| w D_x^{-1/2+3\delta} P_{\geq 8}v \right\|_{L^2} \\
 &\lesssim \|D_x^{-1/2+3\delta} P_{\geq 8}v\|_{L_{tx}^\infty} \|w\|_{L_{tx}^2} \\
 &\lesssim \|v\|_{X_\varepsilon^{1/4,3/4}} \|w\|_{L_{tx}^2} \\
 &\lesssim (\|v\|_{X_\varepsilon^{0,1}} + \|\partial_x v\|_{L_{tx}^2}) \|w\|_{L_{tx}^2}.
 \end{aligned}$$

2.  $|\xi_1| \wedge |\xi_2| < 2^{-7}|\xi|$ . Then, we use that (2.21) holds on the support of  $\eta_{A_\varepsilon}$ . In the subregion  $|\xi_1| \wedge |\xi_2| = |\xi_1|$  we write

$$\begin{aligned}
I^2 &\lesssim \sum_{N \geq 4} \left( \sum_{4 \leq N_1 \leq 2^{-5}N} \left\| \eta_N(\xi) \eta_{A_\varepsilon}(\xi) |\xi| \chi_{\{|\sigma| \sim \max(N_1 N^2, \varepsilon N^4 N_1)\}} \mathcal{F}_x \left( (w P_{N_1} v) \right) \right\|_{X_\varepsilon^{0, -1/2, 1}} \right)^2 \\
&\lesssim \sum_{N \geq 4} \left( \sum_{4 \leq N_1 \leq 2^{-5}N} \|P_{N_1} D_x^{-1/2} v\|_{L_{tx}^\infty} \|\chi_{\{|\xi| \sim N\}} \widehat{w}\|_{L_{\tau, \xi}^2} \right)^2 \\
&\lesssim \sum_{N \geq 4} \|\chi_{\{|\xi| \sim N\}} \widehat{w}\|_{L_{\tau, \xi}^2}^2 \left( \sum_{4 \leq N_1 \leq 2^{-5}N} N_1^{-1/4} \|P_{N_1} D_x^{1/4} v\|_{L_t^\infty L_x^2} \right)^2 \\
&\lesssim \|v\|_{X^{1/4, 3/4}}^2 \|w\|_{L_{tx}^2}^2 \\
&\lesssim (\|v\|_{X^{0, 1}} + \|\partial_x v\|_{L_{tx}^2})^2 \|w\|_{L_{tx}^2}^2.
\end{aligned}$$

Finally, in the subregion  $|\xi_1| \wedge |\xi_2| = |\xi_2|$  we write

$$\begin{aligned}
I^2 &\lesssim \sum_{N \geq 4} \left( \sum_{4 \leq N_2 \leq 2^{-5}N} \left\| \eta_N(\xi) \eta_{A_\varepsilon}(\xi) |\xi| \chi_{\{|\sigma| \sim \max(N_2 N^2, \varepsilon N^4 N_2)\}} \mathcal{F}_x \left( (v P_{N_2} w) \right) \right\|_{X_\varepsilon^{0, -1/2, 1}} \right)^2 \\
&\lesssim \sum_{N \geq 4} \left( \sum_{4 \leq N_2 \leq 2^{-5}N} \|P_{N_2} D_x^{-3/2} w\|_{L_{tx}^\infty} \|\chi_{\{|\xi| \sim N\}} \xi \widehat{v}\|_{L_{\tau, \xi}^2} \right)^2 \\
&\lesssim \sum_{N \geq 4} \|\chi_{\{|\xi| \sim N\}} \xi \widehat{v}\|_{L_{\tau, \xi}^2}^2 \left( \sum_{4 \leq N_2 \leq 2^{-5}N} N_2^{-1/4} \|P_{N_2} D_x^{-3/4} w\|_{L_t^\infty L_x^2} \right)^2 \\
&\lesssim \|w\|_{X_\varepsilon^{-3/4, 3/4}}^2 \|\partial_x v\|_{L_{tx}^2}^2 \\
&\lesssim (\|w\|_{X_\varepsilon^{-1, 1}} + \|w\|_{L_{tx}^2})^2 \|\partial_x v\|_{L_{tx}^2}^2.
\end{aligned}$$

□

Now we are in position to prove the main bilinear estimates :

**Lemma 5.2.**

$$(5.6) \|P_{A_\varepsilon} \partial_x (vw)\|_{X_\varepsilon^{0, -1/2, 1}} \lesssim T^{\frac{1}{4}-} \left( \|w\|_{Y_\varepsilon^0} + \|w\|_{X_\varepsilon^{-1, 1}} \right) \left( \|v\|_{Y_\varepsilon^1} + \|v\|_{X_\varepsilon^{0, 1}} \right),$$

where the functions  $u$  and  $v$  are supported in time in  $] - T, T[$  with  $0 < T \leq 1$ .

*Proof.* First, according to (5.2)-(5.4) and to the support of  $\eta_{A_\varepsilon}$  it suffices to consider

$$I := \left[ \sum_{N \geq 4} \left( \sum_L L^{-1/2} \left\| \eta_L(\sigma) \eta_N(\xi) \int_{\mathbb{R}^2} \sum_{N_1 \wedge N_2 \geq 8} \widehat{P_{N_1} v}(\xi_1, \tau_1) \widehat{P_{N_2} w}(\xi_2, \tau_2) d\tau_1 d\xi_1 \right\|_{L_{\tau, \xi}^2(|\xi| \notin J_\varepsilon)} \right)^2 \right]^{1/2},$$

where  $J_\varepsilon$  is defined in (2.22). We consider different contributions to  $I$ .

1.  $N_1 \wedge N_2 < 2^{-10}(N_1 \vee N_2)$ . Then it holds

$$(1 - 2^{-7})\xi^2 \leq \xi^2 - \xi_1(\xi - \xi_1) \leq (1 + 2^{-7})\xi^2$$

and it is easy to check that  $\Gamma(\xi, \xi_1) \geq 2^{-5}$  as soon as  $|\xi| \notin J_\varepsilon$ . According to (2.13) this ensures that (2.19) holds.

2.  $N_1 \wedge N_2 \geq 2^{-10}(N_1 \vee N_2)$ . Then  $N_1 \sim N_2 \gtrsim N$ .

2.1. The subregion  $|\xi| \notin \left[ \sqrt{\frac{17}{80\varepsilon}}, \sqrt{\frac{2}{5\varepsilon}} \right]$ . In this region, by (2.5) of Lemma 2.1 and duality, we get

$$\begin{aligned} I &\lesssim \sum_{N_1 \wedge N_2 \geq 8, N_1 \sim N_2} \|D_x^{-\frac{1}{4}+} \partial_x (P_{N_1} v P_{N_2} w)\|_{L_t^{\frac{4}{3}+} L_x^{1+}} \\ &\lesssim \sum_{N_1 \wedge N_2 \geq 8, N_1 \sim N_2} T^{\frac{3}{4}-} N_1^{-\frac{1}{4}+} \|\partial_x P_{N_1} v\|_{L_t^\infty L_x^{2+}} \|P_{N_2} w\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{3}{4}-} \|v\|_{L_t^\infty H^1} \|w\|_{L_t^\infty L_x^2}. \end{aligned}$$

2.2. The subregion  $|\xi| \in \left[ \sqrt{\frac{17}{80\varepsilon}}, \sqrt{\frac{2}{5\varepsilon}} \right]$ .

2.2.1 The subregion  $|\xi_1| \wedge |\xi_2| \leq \sqrt{\frac{17}{80\varepsilon}}$ . Since both cases can be treated in the same way, we assume  $|\xi_1| \wedge |\xi_2| = |\xi_1|$ . Then, according to (2.5) and the support of  $\eta_{A_\varepsilon}$  and  $\eta_{B_\varepsilon}$ , we get

$$\begin{aligned} I &\lesssim \sum_{N_1 \wedge N_2 \geq 8, N_1 \sim N_2} T^{\frac{1}{2}-} \|\partial_x (P_{B_\varepsilon} P_{A_\varepsilon} P_{N_1} v P_{N_2} w)\|_{L_{tx}^2} \\ &\lesssim T^{\frac{1}{2}-} \sum_{N_1 \wedge N_2 \geq 8, N_1 \sim N_2} \|P_{B_\varepsilon} P_{A_\varepsilon} \partial_x P_{N_1} v\|_{L_t^4 L_x^\infty} \|P_{N_2} w\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}-} \sum_{N_1 \wedge N_2 \geq 8, N_1 \sim N_2} N_1^{-1/4} \|P_{A_\varepsilon} P_{N_1} v\|_{X_\varepsilon^{1,1/2,1}} \|P_{N_2} w\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}-} \|P_{A_\varepsilon} v\|_{X_\varepsilon^{1,1/2,1}} \|w\|_{L_t^\infty L_x^2}. \end{aligned}$$

2.2.2 The subregion  $|\xi_1| \wedge |\xi_2| > \sqrt{\frac{17}{80\varepsilon}}$ . Then as in the proof of (2.12) in Section 3 we observe that (2.19) holds.  $\square$

To complete the proof of Proposition 5.1 we notice that, similarly to Lemma 2.2, one can easily prove that any solution  $w \in C([0, T]; L^2(\mathbb{R}))$  with  $0 < T < 1$  of (4.8) satisfies

$$(5.7) \quad \|w\|_{X_{\varepsilon,T}^{-1,1}} \lesssim \|w\|_{L_T^\infty H_x^{-1}} + \|v\|_{L_T^\infty H_x^1} \|w\|_{L_T^\infty L_x^2}.$$

Finally, with (5.6) and (5.7) in hand, Proposition 5.1 follows from the classical linear estimates in Bourgain's spaces.  $\square$

Now the second step consists in proving the following estimate :

**Proposition 5.2.** *Let  $0 < \varepsilon < 1$ ,  $w \in C([0, T]; H^1(\mathbb{R}))$  a solution to (4.8) with initial data  $\varphi$  and  $v \in Y_{\varepsilon,T}^1$ . Then it holds*

$$(5.8) \quad \|P_{\mathcal{A}_\varepsilon} w\|_{L_T^\infty L_x^2}^2 \lesssim \|P_{\mathcal{A}_\varepsilon} \varphi\|_{L^2}^2 + (\varepsilon^{1/2} + T^{1/4}) \|w\|_{Y_{\varepsilon,T}^0}^2 \left( \|v\|_{Y_{\varepsilon,T}^1} + \|v\|_{Y_{\varepsilon,T}^1}^2 \right)$$

where the implicit constant is independent of  $\varepsilon$ .

*Proof.* Applying the operator  $P_{\mathbb{C}A_\varepsilon}$  on (4.8) and taking the  $L_x^2$ -scalar product with  $P_{\mathbb{C}A_\varepsilon} w$  we get

$$\begin{aligned} \frac{d}{dt} \|P_{\mathbb{C}A_\varepsilon} w(t)\|_{L_x^2}^2 &= \int_{\mathbb{R}} P_{\mathbb{C}A_\varepsilon} \partial_x (vw) P_{\mathbb{C}A_\varepsilon} w \\ &= \int_{\mathbb{R}} P_{\mathbb{C}A_\varepsilon} \partial_x (w P_{B_\varepsilon} v) P_{\mathbb{C}A_\varepsilon} w + \int_{\mathbb{R}} P_{\mathbb{C}A_\varepsilon} \partial_x (w P_{\mathbb{C}B_\varepsilon} v) P_{\mathbb{C}A_\varepsilon} w \\ &= I_1 + I_2. \end{aligned}$$

Using the following commutator estimate (see for instance [10])

$$\|[P_{\mathbb{C}A_\varepsilon} \partial_x, f]g\|_{L_x^2} \lesssim \|\partial_x f\|_{L_x^\infty} \|g\|_{L_x^2},$$

and integrating by parts, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} P_{B_\varepsilon} v P_{\mathbb{C}A_\varepsilon} w_x P_{\mathbb{C}A_\varepsilon} w + \int_{\mathbb{R}} ([P_{\mathbb{C}A_\varepsilon} \partial_x, P_{B_\varepsilon} v] w) P_{\mathbb{C}A_\varepsilon} w \\ &\lesssim \|\partial_x P_{B_\varepsilon} v\|_{L^\infty} \|w\|_{L_x^2}^2. \end{aligned}$$

By the frequency projections, we easily control  $I_2$  by

$$\begin{aligned} I_2 &\lesssim \varepsilon^{-1/2} \|P_{\mathbb{C}A_\varepsilon} (w P_{\mathbb{C}B_\varepsilon} v)\|_{L_x^1} \|P_{\mathbb{C}A_\varepsilon} w\|_{L_x^\infty} \\ &\lesssim \|P_{\mathbb{C}A_\varepsilon} w\|_{L_x^\infty} \|w\|_{L_x^2} \|v\|_{H_x^1}. \end{aligned}$$

Gathering the above estimates we infer that

$$\frac{d}{dt} \|P_{\mathbb{C}A_\varepsilon} w(t)\|_{L_x^2}^2 \lesssim \left( (\|w(t)\|_{L_x^2} + \|P_{\mathbb{C}A_\varepsilon} w(t)\|_{L_x^\infty}) (\|\partial_x P_{B_\varepsilon} v(t)\|_{L_x^\infty} + \|v(t)\|_{H^1}) \right) \|w(t)\|_{L_x^2}.$$

On the other hand, applying Lemma 3.1 on (4.8) we get

$$\|P_{\mathbb{C}A_\varepsilon} w\|_{L_T^1 L_x^\infty} \lesssim (\varepsilon^{1/2} + T) \|P_{\mathbb{C}A_\varepsilon} w\|_{L_T^\infty L_x^2} + T \|v\|_{L_T^\infty H_x^1} \|w\|_{L_T^\infty L_x^2}.$$

Therefore, integrating in time the next to the last inequality with (4.7) in hand, leads to (5.8)  $\square$

Combining Propositions 5.1 and 5.2 we infer that

$$\|w\|_{Y_{\varepsilon,T}^0}^2 \leq C \|\varphi\|_{L^2}^2 + C (\sqrt{\varepsilon} + T^{\frac{1}{4}}) \|w\|_{Y_{\varepsilon,T}^0}^2 \|v\|_{Y_{\varepsilon,T}^1} \left( 1 + \|v\|_{Y_{\varepsilon,T}^1}^3 \right),$$

which yields the desired result according to (4.6)

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